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BASIC PROBLEMS IN q-HYPERGEOMETRIC FUNCTIONS

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Abstract: Three basic problems on q-hypergeometric functions are presented using Jackson integrals. As an example they are explained in more details in the case of BC_1-type.

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1. Jackson Integrals and q-Hypergeometric Functions

Assume that an element $\alpha \in \mathrm{Hom}(\mathbf{Z}^n, \mathbf{C})$ and a finite set M in $\mathrm{Hom}(\mathbf{Z}^n, \mathbf{Z}) - \{0\}$ are given, then we can define the q-multiplicative function of $t = (t_1, \ldots, t_n) \in X = (\mathbf{C}^*)^n$

$$\Phi(t) = t^\alpha \prod_{\mu \in M} \frac{(a'_\mu t^\mu; q)_\infty}{(a_\mu t^\mu; q)_\infty}$$

for arbitrary $a_\mu, a'_\mu \in \mathbf{C}^*$ ($\mu \in M$) (here we denote $t^\alpha = t_1^{\alpha(\chi_1)} \cdots t_n^{\alpha(\chi_n)}$, $t^\mu = t_1^{\mu(\chi_1)} \cdots t_n^{\mu(\chi_n)}$ with respect to the standard basis $\{\chi_k\}_{1 \le k \le n}$ of $\mathbf{Z}^n$).

In the sequel we follow the references [1], [7] about the terminologies.

Consider the sum over the orbit $[0, \xi\infty]_q = q^{\mathbf{Z}^n} \cdot \xi$ for a fixed $\xi \in X$ and an admissible $\varphi(t)$ as follows:

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$$\int_{[0,\xi\infty]_q} \Phi(t)\varphi(t)\varpi_q = (1-q)^n \sum_{\chi\in\mathbf{Z}^n} \Phi(q^\chi\xi)\varphi(q^\chi\xi) \quad \in \mathbf{C} \tag{1}$$

$$\varpi_q = \frac{d_q t_1}{t_1}\wedge\cdots\wedge\frac{d_q t_n}{t_n}.$$

This is called "Jackson integrals" provided it is convergent and is denoted by $\langle\varphi,\xi\rangle$. If $q^\chi\xi$ lies in a pole of $\Phi(t)\varphi(t)$, it may be replaced by a suitable residue as its regularization. If the sum is divergent, it must be replaced by a suitable contour integral. From now on we call $[0,\xi\infty]_q$ and its regularization "n-dimensional cycle". (1) gives the pairing between a cohomology class of φ in $H^n(X,\Phi,\nabla_q)$ and an n-dimensional cycle, i.e., it gives a dual element of $H^n(X,\Phi,\nabla_q)$. (1) is a quasi-meromorphic function of ξ which is invariant under the q-shift. Hence it can be represented by elliptic theta functions of ξ. Our interest lies in not only q-periodic structures with respect to ξ, but also holonomic q-differecne structures, asymptotic behaviors with respect to the parameters α, a_μ, a'_μ, and connection relations among various asymptotics for the large parameters like $|\alpha| = \sum_{k=1}^n |\alpha(\chi_k)| \to \infty,\ \ |a_\mu|, |a'_\mu| \to 0,\infty$.

Under a suitable genericity condition, one can prove that $H^n(X,\Phi,\nabla_q)$ has a finite dimension, more precisely

$$\dim H^n(X,\Phi,\nabla_q) = \sum_{\{\mu_1,\ldots,\mu_n\}\subset M} [\mu_1,\ldots,\mu_n]^2$$

holds where $[\mu_1,\ldots,\mu_n]$ denotes the determinant of the matrix $(\mu_j(\chi_k))_{j,k}$. For the proof see [7], [12] and the references in them.

In the sequel we shall denote by κ $\dim H^n(X,\Phi,\nabla_q)$.

2. Statement of Problems

Problem 1. Finding explicitly the holonomic q-difference equations satisfied by $\langle\varphi,\xi\rangle$.

Suppose that $\varphi_k(t),\ 1 \le k \le \kappa$, give a basis of $H^n(X,\Phi,\nabla_q)$. The q-shift operators $T_{u_k}, T_{a_\mu}, T_{a'_\mu}$ corresponding to the parameters $u_k = q^{\alpha(\chi_k)}, a_\mu, a'_\mu$ transform $H^n(X,\Phi,\nabla_q)$ into itself. As a consequence, we have the holonomic q-difference equations with the coeffcients of rational functions of $u = (u_k)_k$,

$a_\nu,\ a'_\nu,\ (\nu \in M)$:

$$\begin{cases} T_{u_k}\langle \varphi_j, \xi\rangle = \sum_{l=1}^{\kappa} y_{lj}^{(u_k)} \langle \varphi_l, \xi\rangle, \\ T_{a_\mu}\langle \varphi_j, \xi\rangle = \sum_{l=1}^{\kappa} y_{lj}^{(a_\mu)} \langle \varphi_l, \xi\rangle, \\ T_{a'_\mu}\langle \varphi_j, \xi\rangle = \sum_{l=1}^{\kappa} y_{lj}^{(a'_\mu)} \langle \varphi_l, \xi\rangle. \end{cases}$$

Problem 2. We fix $\xi \in X$. When u, a_μ, a'_μ are at the infinity in the direction ω and $\{\eta_\mu, \eta'_\mu\}$ for each $\mu \in M$, namely, when

$$\alpha = \omega N + \hat{\alpha},\ a_\mu = q^{\eta_\mu N}\hat{a}_\mu,\ a'_\mu = q^{\eta'_\mu N}\hat{a}'_\mu \quad (\omega, \eta_\mu, \eta'_\mu \in \mathbf{Z}^n - \{0\})$$

for fixed $\hat{\alpha},\ \hat{a}_\mu,\ \hat{a}'_\mu$, the asymptotic behaviors of (1) with respect to $N \to \infty$ (N a positive integer) generally can be expressed as

$$\langle \varphi, \xi\rangle \approx C q^{rN(N-1)} \rho^N (1 + O(\frac{1}{N}))$$

for a non-zero pseudo-constant C, a constant $\rho \in \mathbf{C}^*$ and an integer r. It is an interesting problem to evaluate them.

Problem 3. Generally one can determine the κ characteristic cycles corresponding to the given direction $\omega, \{\eta_\mu, \eta'_\mu\}$. If we denote them by $[0, \xi(1)\infty]_q$, $\ldots$, $[0, \xi(\kappa)\infty]_q$, then the Jackson integral (1) over the general $[0, \xi\infty]_q$ can be represented by a linear combination of the integrals over $[0, \xi(k)\infty]_q$ $(1 \le k \le \kappa)$:

$$[0, \xi\infty]_q = \sum_{k=1}^{\kappa} ([0, \xi\infty]_q : [0, \xi(k)\infty]_q)_\Phi \cdot [0, \xi(k)\infty]_q\,,$$

where $([0, \xi\infty]_q : [0, \xi(k)\infty]_q)_\Phi$ are pseudo-constants with respect to $\xi, \alpha, a_\mu, a'_\mu$, and can be described by elliptic theta functions. It is an interesting problem to evaluate the connection coefficients $([0, \xi\infty]_q : [0, \xi(k)\infty]_q)_\Phi$.

In the next section we shall focus our argument on the case of q-hypergeometric functions of BC_1 type.

3. q-Hypergemetric Functions of BC_1-Type

Assume $n = 1$ and let s be a non-negative integer. Take as $\Phi(t)$

$$\Phi(t) = \prod_{k=1}^{2s+2} t^{1/2-\alpha_k} \frac{(qt/a_k;q)_\infty}{(ta_k;q)_\infty}\,, \tag{2}$$

where we put $a_k = q^{\alpha_k}$. This function is of BC_1-type, because $\Phi(t)$ is symmetric with respect to the inversion σ $(t \to 1/t)$:

$$b(t) = \Phi(qt)/\Phi(t) = \Phi(1/(qt))/\Phi(1/t)$$

σ acts on $H^1(X, \Phi, \nabla_q)$ as an endomorphism. We are only interested in its skew-symmetric part $H^1_{skew}(X, \Phi, \nabla_q)$. We have the basic identity

$$\int_{[0,\xi\infty]_q} \Phi(t)\{\nabla_q\psi(t) - \nabla_q\psi(1/t)\}\varpi_q = 0\,, \tag{3}$$

where $\nabla_q\psi(t) = \psi(t) - b(t)\psi(qt)$ for an admissible rational function $\psi(t)$. The dimension of $H^1_{skew}(X, \Phi, \nabla_q)$ is equal to s and one can choose as a basis the representatives $\varphi_k = t^k - t^{-k}$ $(1 \le k \le s)$ (see [5], [6] for more details).

The holonomic q-difference equations with respect to $a_1, \dots, a_{2s+2}$ are given as follows (see [3],[6]) :

$$T_{a_k}(J_j) = -\Big(a_k + \frac{1}{a_k}\Big)J_j + J_{j+1} + J_{j-1} \quad (1 \le j \le s-1),$$

$$T_{a_k}(J_s) = -\Big(a_k + \frac{1}{a_k}\Big)J_s + J_{s-1} + \sum_{r=1}^{s}(-1)^{s-r}\frac{\varepsilon_{s-r+1} - \varepsilon_{s+r+1}}{1 - \varepsilon_{2s+2}}J_r\,,$$

where J_k denotes $\langle\varphi_k, \xi\rangle$, $J_0 = 0$ and ε_k denotes the elementary symmetric polynomial of degree k in $a_1, \dots, a_{2s+2}$. One can prove that the above q-difference equations have the fundamental matrix solution $Y = Y(a_1, \dots, a_m)$ such that $Y/\prod_{k=1}^m \vartheta(a_k;q)$ is holomorphic at its origin, where $\vartheta(x;q)$ denotes the elliptic theta function $(x;q)_\infty(q/x;q)_\infty(q;q)_\infty$ (see [3]).

One can choose as a dual basis of $H^1_{skew}(X, \Phi, \nabla_q)$ the cycles $[0, a_k\infty]_q$, where k moves over a subset of s indices $K \subset \{1, 2, \dots, 2s+2\}$. As for the connection formula among a general $[0, \xi\infty]_q$ and $[0, a_k\infty]_q$ we have (see [10])

$$([0,\xi\infty]_q : [0, a_k\infty]_q)_\Phi = \frac{\Theta(\xi)}{\Theta(a_k)} \prod_{\substack{j\in K\\ j\neq k}} \frac{\vartheta(a_j\xi;q)\vartheta(a_j/\xi;q)}{\vartheta(a_ja_k;q)\vartheta(a_j/a_k;q)}\,,$$

where

$$\Theta(\xi) = \xi^{s-\sum_{k=1}^{2s+2}\alpha_k} \frac{\vartheta(\xi^2;q)}{\prod_{k=1}^{2s+2}\vartheta(a_k\xi;q)}\,.$$

In case where $s = 1$, (1) reduces to Bailey's ${}_6\psi_6$- formula (see [8], [9], [13]). In case where $s = 2$, it reduces to Askey-Wilson polynomials and their Stieltjes transforms with respect to the variable z by taking $a_j q^{n/2}$ $(1 \leq j \leq 4)$, $a_5 = zq^{-n/2}$, $a_6 = z^{-1}q^{-n/2}$ $(n = 0, 1, 2, \ldots)$ instead of a_j $(1 \leq j \leq 6)$ respectively (see [4]).

Remark. The Jackson integrals corresponding to (2) can also be generalized to multivariable cases. They satisfy holonomic q-difference equations. However we have not yet succeeded in getting explicit formulae (see [5], [6]).

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