International Journal of Pure and Applied Mathematics
Volume 49 No. 4 2008, 569-575

A NON-LOCAL CONSERVATION-LAW NUMERICAL SCHEME WITH FIDELITY

A. Louise Perkins[1 §], Alice Qiao[2], Farnaz Zand[3]
[1,2,3]Department of Mathematics
University of Southern Mississippi
730 East Beach Boulevard Long Beach, MS 39560, USA
[1]e-mail: louise.perkins@usm.edu

Abstract: In order to control nonlinear frequency and amplitude errors, we allow the Courant number (σ) to vary according to the maximum velocity at each time step. The method requires significant *a priori* work. However, it is computationally efficient.

AMS Subject Classification: 26A33
Key Words: nonlinear, frequency space, adaptive FDM

1. Introduction

To apply von Neumann stability analysis, assume the difference equation has solutions of the form $A^k e^{im\omega}$, where m is an arbitrary wave, A is the amplification factor and ω is the spatial component. Here we write the amplification factor in the form $g = |g|e^{i\phi}$ so that our dispersion exponent is explicit, as is our scalar numerical dissipation.

We compare our method to five existing schemes for the 1-D Burgers equation, Euler/Upwind FDS, Leapfrog scheme, Lax-Friedrichs scheme, Lax-Wendroff (c.f. Strikwerda [7]), and the Wavenumber Extended Upwind-Biased scheme (c.f. Li [2]),

$$u_x = \frac{1}{\Delta x}(-0.055453u_{m-3} + 0.360600u_{m-2} - 1.221201u_{m-1} + 0.554534u_m + 0.389400u_{m+1} - 0.027880u_{m+2})\,.$$

Received: August 14, 2008

§Correspondence author

Then approximate the first derivative in time similar to the E/U FDS. It has the same dispersion characteristics as a centered difference scheme of order five plus a correction that reduces the numerical dissipation.

To construct high fidelity numerical solutions, we begin with the Frequency Accurate FDS (Orlin and Perkins [6]) scheme to insure that our phase speeds are nearly correct. We then include a filter to remove high frequency accumulation. Lastly we add a step that controls amplitude growth. For the nonlinear inviscid hyperbolic Burgers equation we use J equal spatial increments, Δx, on a domain of length L, indexed by 1, 2, $\ldots, J$. The time increment is Δt. For this grid, the spatial and temporal Nyquist intervals are $\left[\frac{-1}{2}\Delta x, \frac{l}{2}\Delta x\right]$ and $\left[\frac{-1}{2}\Delta t, \frac{1}{2}\Delta t\right]$, respectively, which corresponds to $[-\pi, \pi]$ in a scaled frequency domain. Outside this frequency band, folding and aliasing will occur.

For the Frequency Accurate FDS, developing a simulation is done using separate spatial (Orlin et al [4]) and temporal (Orlin et al [5]) frequency accurate approximations. The spatial difference operator is defined as,

$$\overline{\Delta}_x(U_m^n) = \left(\frac{1}{\nu}\overline{\Delta}_x^{[+]} + \frac{\nu-1}{\nu}\overline{\Delta}_x^{[-]}\right). \quad (1)$$

The ν is an "upwind" weighting factor. The average value temporal difference operator is defined in a similar fashion:

$$\bar{\Delta}_t(U_m^n) = \frac{1}{\Delta t}\left[U_m^{n+1} - \frac{1}{\mu}\sum_{j=0}^{N}\left(\theta_j^{[+]}U_{m+j}^n + (\mu-1)\theta_j^{[-]}U_{m-j}^n\right)\right]. \quad (2)$$

An approximation to the continuous advection equation can now be written. Solving for U_m^{n+1} yields the FDS given in Orlin [3] for the approximation,

$$U_m^{n+1} = \frac{\sigma(2-\nu)}{\nu}U_m^n + \sum_{j=0}^{N}\left[\left(\frac{1}{\mu}\theta_j^{[+]} - \frac{\sigma}{\nu}\psi_j^{[+]}\right)U_{m+j}^n + \left(\frac{\mu-1}{\mu}\theta_j^{[-]} + \frac{\sigma(\nu-1)}{\nu}\psi_j^{[-]}\right)U_{m-j}^n\right] \quad (3)$$

where the coefficients μ and ν are yet to be determined. Transforming to the frequency domain nets

$$\hat{\bar{\Delta}}_t(\hat{U}) + c\hat{\bar{\Delta}}_x(\hat{U}) = 0. \quad (4)$$

Then, taking the difference between the transform of the true advection equation and the transform of our approximation, we form an error gain in the frequency domain. This is a function of σ. Next, specify $N-1$ arbitrary fitting frequencies within $[0, \pi]$ and force the error gain to vanish at these frequencies. This is done in such a way as to insure convergence and consistency.

First, we find the weighting coefficients to approximate u_x; then find the weighting coefficients for u_t (Orlin [3]). Finally, we cast the FDS for the advection equation in operator form for later use

$$U_m^{n+1} = \Delta_t(U_m^n) + \sigma\Delta_x(U_m^n) = \mathsf{T}(U_m^n) - \sigma\mathsf{L}(U_m^n). \qquad (5)$$

We computed fittings over a range of Courant numbers (advective speeds) and stored the undetermined coefficients in an amplitude dependent look-up table. This provides an efficient way to construct a dynamic approximation that may be phase accurate even for nonlinear equations. Consider the lagged nonlinear interaction of two arbitrary components λ_l and λ_m, with corresponding wave number subscripts. It can be seen that

$$\begin{aligned} u_j^n\,(u_j^n - u_{j-1}^n) &= \sum_{k_l} A^n e^{i\frac{2\pi k_l j}{J}} (\sum_{k_m} A^n e^{i\frac{2\pi k_m j}{J}} + \sum_{k_m} A^n e^{i\frac{2\pi k_m (j-1)}{J}}) \\ &= \sum_{k_l}\sum_{k_m} A^n e^{i\frac{2\pi(k_l+k_m)j}{J}} (A^n + A^n e^{i\frac{2\pi(k_l-k_m)}{J}}). \qquad (6) \end{aligned}$$

The interaction of λ_l and λ_m will not be interpreted correctly on a grid when $k_l + k_m > J/2$, since the smallest wavelength recognized by the grid system is $2\Delta x$ which corresponds to the largest wave number $J/2$.

We design our frequency and amplitude accurate FDS to minimize damping of the amplitudes at low frequencies so as to be close to the true solution, but large damping at high frequencies so as to avoid nonlinear instability. In other words, we design a FDS with built-in high frequency filtering.

During the design process, increasing the value of ν, which is a fitting parameter, results in increasing the magnitude of the real part of the complex response of the approximation, which controls the amount of dissipation (artificial viscosity).

We use a simple centered average (or mean) filter:

$$Y(t) = \frac{1}{2N+1}\sum_{n=-N}^{N} X(t+n\Delta t) \;=\; \bar{X}.$$

Here it is immediately apparent that the weighting coefficient for each X_n is

$$w_n = \frac{1}{2N+1}, \;\text{ for } -N \le n \le N.$$

$$R(\kappa) = \sum_{n=-N}^{N} w_n e^{in\kappa} = \frac{1}{2N+1}\sum_{n=-N}^{N} e^{in\kappa} = \frac{1}{2N+1}\frac{\sin(2N+1)\kappa/2}{\sin\kappa/2}\,. \qquad (7)$$

Note that $R(0) = 1$ gives unity gain at zero frequency.

Consider the non-linear Burgers equation as a linear advection equation locally and temporally in the limited sense:

$$\lim_{(x,t)\to(x_a,t_a)} u_t(x,t) + u(x,t)\, u_x(x,t) = u_t(x_a,t_a) + c\, u_x(x_a,t_a) \tag{8}$$

where (x_a, t_a) is an arbitrary point in space and time, and the propagation speed is $c = u(x_a, t_a)$.

We know the velocity u_j^n at each grid point j, and each time step n, we compute a Courant number for each time step n at each j, i.e.,

$$\sigma_j^n = \frac{u_j^n\, \Delta t}{\Delta x} \tag{9}$$

For each time step and location, the σ may vary. Notice that $\vec{\sigma}$ is dependent on both space and time, but even though σ varies due to u_j^n, Δt is the same for all the grid points at the same time step.

We have a vector $\vec{\sigma}$ in space whose components σ_j^n depend on the velocity u_j^n. Fitting coefficients are calculated and stored in look-up tables for a range of σ values. After we assemble all the needed coefficients, we update u_j^n to u_j^{n+1}.

The look-up tables are represented by simple interpolation curves. We choose $\Delta\sigma = 0.001$. Since any single constant ν in our fitting relation cannot give the same good performance for each different σ, a vector ν is used instead, where each element in ν corrensponds to and element in σ. We next construct an empirical relationship between ν and σ so as to achieve the frequency response previously described:

$$\vec{\nu} ::= 7.0\, \vec{\sigma} + 2 \tag{10}$$

This provides virtually no damping in a large area of low frequencies while providing large damping in high frequencies. It also has a response whose phase velocity factor for different σ is close to unity at these lower frequencies.

One way to view adjusting σ, is as a change in the time increment Δt. Since we want to have the same time step for all grid points, we select the maximum allowable time step for all grid points (we normalize $\vec{\sigma}$). The normalization is an improvement in accuracy. For hyperbolic partial differential equations it is best to take σ close to the stability limit, i.e. 1, to keep the dispersion and dissipation small (Strikwerda [7]) – σ small will suffer larger phase errors than σ near unity (Crowley [1], Tam and Webb [8]). Hence the scheme proposed is a near DRP scheme for the frequencies of interest.

We can view this as a two-step Conservative Dynamic filter (CD filter). It achieves nonlinear stability by reducing both spatial and temporal high frequencies and adjusting the gain factor. Thus, it corrects amplitude errors after the phase has been advanced accurately.

Next we filter time:

$$\bar{u} = \frac{u^{n-1} + u^n}{2}. \tag{11}$$

The phase and the frequency of the filtered wave do not change. The amplitude of the wave is reduced by $\sin(\frac{1}{2}\omega\Delta x)$ quantity squared. The filtered amplitude will remain essentially unchanged only for very high-resolution grids – the filter will always damp the signal. To correct this unwanted damping, we introduce an adjustable gain factor G as an optional post-processing step. This post-processing step has merit in its own right. There is a need after each time step to correct biased errors, and this is an opportunity to do so.

We use a gain factor G as a dynamic parameter, which we adjust to give a "proper" gain after each filtering step. The adjusted value becomes

$$\tilde{u} = G\bar{u} \tag{12}$$

Clearly, this is a scaling. The shape of the scaled function $\tilde{u}$ is the same as that of $\bar{u}$. The gain factor may be greater than 1. Yet using a G whose value is slightly greater than 1 can compensate for the unwanted damping in low frequencies without becoming unstable. Consider $u_t + uu_x = \gamma u_{xx}$. For uniform convergent u, to keep energy constant we need

$$\frac{d}{dt}\int_{-\infty}^{\infty} u dx = \int_{-\infty}^{\infty} u_t dx = \int_{-\infty}^{\infty} \gamma u_{xx} - uu_x dx = [\gamma u_x - \frac{1}{2}u^2]_{-\infty}^{\infty} = 0\,. \tag{13}$$

Let the initial area of the Burgers wave at time $t = 0$ be A^0, and for time step n, A^n. Define a relative error of the residual wave area as $r^n = \frac{(A^n - A^0)}{A^0}$. We see that

$$A^n = A^0\, r^n + A^0 = A^0(1 + r^n)\,.$$

Multiply each individual value of $f(x)$ in the grid by the gain factor G^n to compute the post-processed energy. Combine this with the above equations nets $A^n = \frac{1}{(1+r^n)}$.

G^n is evaluated according to the relative error of the residual wave area at the current time step. If $r^n = 0$, then $G^n = 1$ and no adjustment need to be done. If $r^n < 0$, then $G^n > 1$ and we supply greater amplitude to compensate for the unwanted damping. If $r^n > 0$, then $G^n < 1$ and we supply additional damping. The adjustment of G is done dynamically at each time step. The scheme is amplitude correct across the spectrum becuase the gain factor G is independent of frequencies. However, we also constructed the gain factor as a

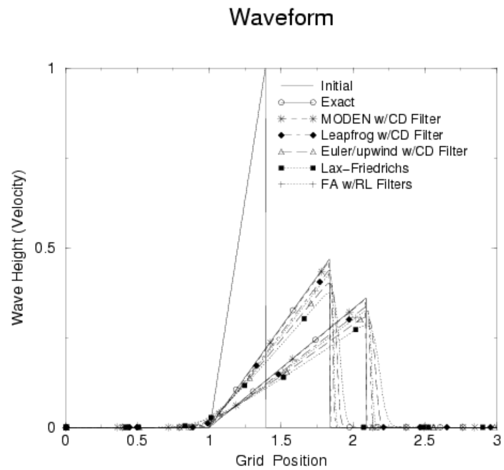

Figure 1: The results of the tested finite difference schemes

function of frequency, i.e. $G(\omega)$. However, since the process involves the Fourier transform, it is not efficient.

In our Burgers solutions (Figure 1), the PSD of the highest spatial frequency, π, is less than 10^{-16}. The high frequency noise has been tremendously suppressed. It is obvious that the amplitude of the computational mode is controlled. The behavior at around time step 340 is not well understood. We conjecture that a nonlinear instability began to develop, but was subsequently suppressed. As we integrate in time, the values of the relative wave area (error) appear to be random numbers at machine precision. Taken together with frequency accurate methods, we conclude that we have avoided accumulating LTE. We attribute this to having both the phase and amplitude nearly exact.

The Euler/upwind, the Leapfrog, and the fourth-order wavenumber-extended upwind-biased schemes failed to simulate the up-triangle Burgers' equation realistically, while the Lax-Friedrichs scheme introduced large dispersion and dissipation. The improvements mentioned in this paper partially corrected these failings even in these methods.

References

[1] P.W. Crowley, Numerical advection experiments, *Monthly Weather Review*, **96**, No. 1 (1968), 1-11.

[2] Y. Li, Wavenumber-extended high-order upwind-biased finite-difference schemes for convective scalar transport, *Journal of Computational Physics*, **133** (1997), 235-255.

[3] A.P. Orlin, *Frequency Accurate Finite Difference Approximations*, Ph.D. Dissertation, University of Southern Mississippi, Hattiesburg, MS, May (1996).

[4] A.P. Orlin, A.L. Perkins, G.W. Heburn, A frequency accurate spatial derivative finite difference approximation, *Numerical Methods for Partial Differential Equations*, **13**, No. 5 (1997), 549-559.

[5] A.P. Orlin, A.L. Perkins, G.W. Heburn, A frequency accurate temporal derivative finite difference approximation, *Journal of Computational Acoustics*, **5**, No. 4 (1997), 371-382.

[6] A.P. Orlin, A.L. Perkins, A frequency accurate finite difference scheme for burgers equation, *Journal of Computational Acoustics*, **6**, No. 4 (1998).

[7] C.J. Strikwerda, *Finite Difference Schemes and Partial Differential Equations*, Wadsworth and Brooks, Pacific Grove, California (1989).

[8] K.W.C. Tam, J.C. Webb, Dispersion-relation-preserving finite difference schemes for computational acoustics, *Journal of Computational Physics*, **107** (1993), 262-281.