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GRAVITY COUPLED TO YANG-MILLS AND THE FIVE DIMENSIONAL GROUP MANIFOLD SCENARIO

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Abstract: The main purpose standing behind the group manifold approach to gravity and supergravity theories, is the need for describing forces of nature by means of non-Riemannian geometries, and to examine their physical contents in terms of a vierbein formalism. Motivated by the close relationship that is connecting quantum gravity and Yang-Mills theory in the non-perturbative strings theory framework, we have extended the group manifold geometrical scenario by introducing a general gauge group $\mathcal{G}$. Although we are still deeply rooted in Regge's ideas that led to the group-manifold approach, our work has had its own particularities: here dynamics is controlled by geometry in the sense that first curvatures and Bianchi identities were established and then the Lagrangian is worked out. Using the basic tools of exterior forms and exterior derivatives, gravitation and its extension as a Yang-Mills theory is then described on a group manifold G. G has the same relationship to the Poincaré group, as curved spacetime does to Minkowski spacetime, except that the existence of a metric is now replaced by assumptions about the vierbein. In this paper, we synthesize the main features of the extended geometrical scenario that we have established and discuss perspectives for the nearest future.

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1. Gravity and its Extensions

As it is well known, the general theory of relativity may be characterized by some basic assumptions: (i) the metric tensor is symmetric ($G_{\mu\nu} = G_{\nu\mu}$); non symmetric theories (Einstein and Kaufman [10]), (Moffat [15]) have been showed to be unsound (Hammond [12]); (ii) $\Delta_\sigma G_{\mu\nu} \equiv Q_{\sigma\mu\nu} = 0$ (Q = non metricity tensor); by assuming that this vanishes, we are assumed that lenghts of vector upon parallel transport are invariant; (iii) $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$ (the affine connection is symmetric); (iv) $\delta\int -\sqrt{G}Rdx^4 = 0$ (Hilbert showed that the Einstein Vacuum field equations could be derived from a Variational Principle by considering variations with respect to the metric tensor). For a non-symmetric connection, the antisymmetric part is defined as torsion T:

$$T^\sigma_{\mu\nu} \equiv \Gamma^\sigma_{[\mu\nu]} = \frac{1}{2}({\Gamma^\sigma}_{\mu\nu} - \Gamma^\sigma_{\nu\mu}); \tag{1}$$

$$\Gamma^\sigma_{\mu\nu} = \left\{{\sigma \atop \mu\nu}\right\} + 2T^\sigma_{\mu\nu} + T^\sigma_{\nu\mu} \Rightarrow \Gamma^\sigma_{\mu\nu} = \left\{{\sigma \atop \mu\nu}\right\} \Leftrightarrow T^\sigma_{\nu\mu} = 0; \tag{2}$$

$$T^\sigma_{\nu\mu} = Torsion\,.$$

Elie Cartan [6] was the first to relax the third assumption and introduced torsion. Later many others like Hehl et al [13], Kibble [14], Regge [9],Hammond [12], Borges and Masalskiene [4], Borges [2], [3] and Borges et al [5], worked on theories with a non-symmetric connection. Torsion may be physically considered as the density of intrinsic angular momentum (Hehl et al [13]), on even gauge potential in local gauge theories of gravity (Hammond [12]) through the use of second order formalism; non supersymmetry used. In Regge et al [8], as in Borges [1] through a first order formalism in superspace, it is showed the role of torsion as a propagating field; in that work torsion also appears as constituent of a multiplet, called "Generalized curvature" Θ^A:

$$\Theta^A = (\Theta^{ig}, \Theta^i) \tag{3}$$

being $\Theta^{ig} = d\omega^{ig} - \omega^{ik} \wedge \omega^i_k$ the curvature and $\Theta^i = dv^i - \omega^{ig} \wedge v_g$ the torsion ($\omega^{ig} = \omega^{ig}_\mu dx^\mu$ is the cartan connection on the manifold; $v^i = v^i_\mu dx^\mu$ is the vierbein).

Besides the physical interpretation for torsion, Borges [2] has extended gauge formulation of gravity, in the approach, popularly called "Group manifold approach to gravity and supergravity theories". Essentially, the main purpose of the group manifold approach has been of finding out a formulation for supergravity theories where the group structure would appear in a simple way, making the symmetries properties more transparent and from which all the fields arise as constituent of only one geometrical entity. Motivated by this approach, we have worked in the construction of a geometric structure for an extended five dimensional gravity theory of Yang-Mills type, that will be briefly exposed in the next section. To conclude, we will discuss some features for the next steps to be carried on.

2. Geometric Formulation of Extended Cartan's Gravity

In the last years a geometric structure for the N=2, d=5 Yang-Mills theories on the group manifold has been formulated and discussed by Regge et al [8], Castellani et al [7] and Borges [1], [2]. These theories may be regarded as an embedding of Einstein's theory of gravity in a broader framework. The internal symmetries (gauge groups of Yang-Mills) and the space-time symmetries (gauge groups of Poincaré) are unified in an algebraic structure. In this way also the internal symmetries show a geometrical feature that extend the geometrical interpretation of pure gravity. We are going to be reporting the main steps in the construction of this scenario.

The first step in the construction of the theory is the choice of the group on which the theory is to be formulated. The group is determined from the d=5 supergravity, since the geometrical Supersymmetric Yang-Mills N=2, d=5 can immediately be coupled to the N=2, d=5 supergravity. In the case of the supergravity, one can deduce the nature of the group upon analysis of the field content of the theory. The group is characterized by the dual generations of the 1-forms $\bar{\omega}^{ab}$ (SO(1,4) Lorentz connection with 10 parameters), V^a (vierbein associated to the graviton with five parameters), B (spin-1 field with five parameters) and Ψ_A (gravitini with four parameters); it is therefore a 24-parameter group. The group can then be identified as SU(2,2/1) or one of its contractions. Thus, the group manifold G of this theory is given by the direct product between the general gauge group $\mathcal{G}$ and the graded Poincaré supergravity group:

$$G = \mathcal{G} \otimes \overline{SU(2,2/1)}. \tag{4}$$

In the supergravity case it is shown that there exists a gauge invariance of the theory with respect to $H = SO(1,4) \otimes U(1)$. Then, the coupling between Cartan's gravity theory in 5 dimensions (ECGT-5) and supergravity acquires a bundle strucutre, where $H^{'} = \mathcal{G} \otimes SO(1,4) \otimes U(1)$ will be the fibre, and the quotient space $G/H^{'}$, the base space of the principal fibre bundle.

Next, the set of curvatures of group manifold is formulated considering the supergravity curvatures established by Regge et al [8], F the curvature associated to the gauge group $\mathcal{G}$ and the covariant derivatives of the spinorial field λ_A (related to the Dirac equation), scalar field σ and the internal curvatures associated to Yang-Mills $F_{ab} = \frac{1}{2}(\partial_a A_b - \partial_b A_a) = \partial_{[a}A_{b]}$. For the theories on a group manifold an important property must be considered in setting the curvatures too: the rheonomy. This property is equivalent to the possibility of developing all the dynamics in space-time, giving the theory a physical meaning. Besides that, the factorization is considered too, which is associated to the use of the restricted basis, V^A and Ψ_A. With this considerations and following other criteria about degreee of the forms, Lorentz covariance, dimensional analysis and others described in Borges [1] a general hypothesis of the expression for the curvatures is formulated by:

$$\begin{aligned} F = dA &= F_{ab}V^a \wedge V^b + ic\bar{\lambda_A} \wedge \Gamma_m \Psi_A \wedge V^m \\ &+ id\epsilon_{AB}\bar{\lambda_A} \wedge \Gamma_m \Psi_B \wedge V^m + if\sigma \wedge \bar{\Psi_A} \wedge \Psi_A \\ D\lambda_A &= \Lambda_{mA} \wedge V^m + igF_{ab} \wedge \Sigma^{ab}\Psi_A + h\phi_a \wedge \Gamma^a \Psi_A \\ &+ itF_{ab} \wedge \epsilon_{AB}\Sigma^{ab}\Psi_B + Z\phi_a \wedge \Gamma^a \epsilon_{AB}\Psi_B; \\ D\sigma &= \phi_a \wedge V^a + ik\bar{\lambda}_A \wedge \Psi_A + il\epsilon_{AB}\bar{\lambda}_A \wedge \Psi_B; \\ DF^{ab} &= G^{ab}_m \wedge V^m + in\bar{\Lambda}^{[a}_A \wedge \Gamma^{b]}\Psi_A + ip\epsilon_{AB}\bar{\Lambda}^{[a}_A \wedge \Gamma^{b]}\psi_B\,. \end{aligned} \tag{5}$$

Through the analysis of the Bianchi identities it was possible to determine a compatible system of equations for the parameters of the theory, this means that the hypothesis made for the curvatures are acceptable. Thus, the generalized curvature to ECGT-5 is defined by the following multiplet:

$$\Theta_A \equiv (\Phi_A, R_B), \tag{6}$$

where $\Phi_A \equiv (F, D\lambda_A, D\sigma, DF_{ab})$, and R_B the "supergravity curvature multiplet". F is the "internal curvature" associated to the gauge group of Yang-Mills $\mathcal{G}$, and $D\lambda_A, D\sigma, DF_{ab}$, are respectively the covariant derivatives of λ_A, σ and F_{ab}.

After the explicit determination of the curvatures by means of the Bianchi identities, the next step was to build up the geometric action on the group manifold G. The procedure to obtain the Lagrangian used is an application of the general scheme provided by Castellani, D'Auria and Fré [7].

In general, the construction of a Lagrangian for supersymmetric theories is a complicated task. Some basics requirements must be taken into account as preliminary "precautions":

(i) the action is an integral of 5-forms developed on an arbitrary hypersurface M5 (= 5 dimensions) immersed in the whole manifold G;

(ii) the action must be stationary relating to the variations of fields and to the variations of the hypersurface M5;

(iii) the 5-form Lagrangian is gauge invariant under the group $H' = \mathcal{G} \otimes SO(1,4) \otimes U(1)$, with a general form given by $L = \Delta_A + \Theta_A \nu^A + \Theta_A \wedge \Theta_B \nu^{AB} + ...$, where $\Delta_A, \nu^A, \nu^{AB}$ are polynomials with constant coefficients and Θ_A, Θ_B are the generalized curvatures.

Then the Lagrangian L is worked out with all the fields present in the theory and $A, \lambda_A, \sigma, F_{ab}, \phi^a, V_a, \Psi_A$, in first order form. The action A_c for ECGT-5, taken in account all the basic conditions here reported and the fields in first order version (Borges [3]) is the following:

$$
\begin{aligned}
A_C = \int_{M5} \Bigg(& dAF^{ab}V^c \wedge V^d \wedge V^e \varepsilon_{abcde} - \left(\frac{1}{20}\right) F_{ab}F^{ab}V^i \wedge V^j \wedge V^k V^l \wedge V^m \varepsilon_{ijklm} \\
& + \left(\frac{1}{10}\right) \phi^a \phi_a V^i \wedge V^j \wedge V^k V^l \wedge V^m \varepsilon_{ijklm} - \phi^a d\sigma V^b \wedge V^c \wedge V^d \wedge V^e \varepsilon_{abcde} \\
& + \left(\frac{1}{4}\right) \lambda_A \Gamma^a D \lambda_A V^b \wedge V^c \wedge V^d V^e \varepsilon_{abcde} - iF_{ab} \bar{\lambda}_A \Gamma_m \Psi_A V_m \wedge V^c \wedge V^d V^e \varepsilon_{abcde} \\
& - iF_{ab} \bar{\Psi}_A \Psi_A V^c \wedge V^d \wedge V^e \varepsilon_{abcde} + \left(\frac{1}{2}\right) i\phi^a \bar{\lambda}_A \Psi_A V^b \wedge V^c V^d \wedge V^e \varepsilon_{abcde} \\
& - 6dA \bar{\lambda}_A \Sigma^{ab} \Psi_A V^a \wedge V^b - 2(d\sigma) \bar{\lambda}_A \Sigma^{ab} \Psi_A V^c \wedge V^e \varepsilon_{abcde} \\
& - 6i\sigma \bar{\lambda}_A \Sigma^{ab} \Psi_A \bar{\Psi}_B \Psi_B V^a \wedge B^b - \left(\frac{1}{8}\right) i\bar{\lambda}_A \Sigma^{ab} \lambda_A \bar{\Psi}_B \wedge \Psi_B \wedge V^c \wedge V^d \wedge V^e \varepsilon_{abcde} \\
& - \left(\frac{1}{4}\right) i\bar{\lambda}_A \lambda_B \delta_{AC} \bar{\Psi}_C \Sigma^{ab} \Psi_B \wedge V^c \wedge V^d \wedge V^e \varepsilon_{abcde} \\
& + \left(\frac{3}{4}\right) i\bar{\lambda}_A \Sigma^{ab} \lambda_A \bar{\Psi}_B \Gamma^c \Psi_B \wedge V_a \wedge V_b \wedge V_c \\
& + \left(\frac{3}{4}\right) i\bar{\lambda}_A \Gamma^c \lambda_B \delta_{AC} \bar{\Psi}_c \Sigma^{ab} \Psi_B \wedge V_c \wedge V_a \wedge V_b \\
& - 6idA\sigma \left(\bar{\Psi}_A \Gamma^m \Psi_A\right) \wedge V_m - 3\sigma^2 \bar{\Psi}_A \Gamma^m \Psi_A \wedge \bar{\Psi}_B \wedge \Psi_B \wedge V_m \\
& + \left(\frac{3}{2}\right) iA \wedge dA \wedge \bar{\Psi}_A \wedge \Psi_A \Bigg). \qquad (7)
\end{aligned}
$$

The joint action (Ferrara et al [11]) for the system is obtained by the simple addition of the supergravity action A_S (Regge [8]) to the "Cartan's action" A_C for ECGT-5.

3. Conclusions and Perspectives

Following the basic philosophy of the Group Manifold Approach we have been worked in the construction of an alternative gravity theory of the Yang-Mills type over a curved group manifold. In this theory, the space-time is presented as a hypersurface embedded in this geometrical background. The dynamics is controlled by geometry in the sense that first the curvatures and Bianchi identities were established and then the Lagrangian worked out. The results obtained was formulated in the language of differential forms combined with the exterior product, which are more compact than the usual tensor calculus.

Now, through the variational principle we are investigating how to calculate all the motion field equations. To find the lagrangian we assume that the action is stationary relating to the variations of fields and to the variations of the hypersurface M_5. It implies that the equations of motion interpreted in differential forms and resulting from the variations of the Lagrangian related to the fields of the theory will have validity on the whole manifold. We assume that lagrangian is not trivial too. That means that if curvatures of supergravity are zero, then the equations of motion must be satisfactorily worked out, having as content both the curvature of the gauge group $\mathcal{G}$ and of the covariant derivatives $D\lambda_A, D\sigma, DF_{ab}$. It is our hope to present some new results in a forthcoming communication.

References

[1] M.F. Borges, Bianchi identities for an $N = 2$, $d = 5$ supersymmetric Yang-Mills theory on the group manifold, *Journal of Geometry and Physics*, **20** (1996), 142-148.

[2] M.F. Borges, Geometrical Lagrangian for a supersymmetric Yang-Mills theory on the group manifold, *Mathematical Physics, Analysis and Geometry*, **5**, No. 4 (2002), 301-318.

[3] M.F. Borges, Cartan's emblematic legacy: Roots to a contemporany five-dimensional gravity theory, *International Journal of Mathematical Sciences*, **3**, No. 2 (2004), 495-518.

[4] M.F. Borges, S.R.M. Masalskiene, Geometric extended gravity theory of Yang-Mills type, In: *Proceedings of the Eight Marcel Grossmann Meeting on General Relativity*, Gravitation and Field Theories, Jerusalem, 22-27, June 1997, World Scientific (1999), 496-499.

[5] M.F. Borges, J.A.F. Roveda, S.R.M.M. Roveda, Gravity as an extended Yang-Mills theory on the group manifold, *International Journal of Applied Mathematics*, **21**, No. 1 (2008), 117-128.

[6] E. Cartan, Sur le variétés a connexion affine et la théorie de la relativité généralisée, *Annales de l'Escole Normale Supérieure*, **40** (1923), 325; **41** (1924), 1; Reprinted in *Oeuvres Complétes*, **3**, Gauthier-Villars, Paris (1995).

[7] L. Castellani, R. D'Auria, P. Fré, Supergravity and superstrings. A geometric perspective, *Mathematical Foundations*, **1**, World Scientific, Singapore (1991).

[8] R. D'Auria, P. Fré, E. Maina, T. Regge, Geometrical first order supergravity in 5 space-time dimensions, *Annals of Physics*, **135**, No. 237 (1981).

[9] R. D'Auria, P. Fré, E. Maina, T. Regge, A new group theoretical technique for the analysis of Bianchi identities and its application to the auxiliary field problem of $d = 5$ supergravity, *Annals of Physics*, **139** (1982), 93. 0

[10] A. Einstein, B. Kaufman, Algebraic properties of the field in the relativity theory of the asymmetric field, *Ann. of Math*, **21**, No. 59 (1954), 230-244.

[11] S. Ferrara, F. Gliozzi, J. Scherck, P.V. Nieuwenhuizen, Matter couplings in supergravity theory, *Nuclear Physics*, **B 117** (1978), 333.

[12] R. Hammond, Spin Gravity, *General Relativity and Gravitation*, **31**, No. 2 (1999), 233-271.

[13] F.W. Hehl, P. Von der Heyde, G.D. Kerlick, J.M. Nester, General relativity with spin and torsion: Foundations and prospects, *Review of Modern Physics*, **48**, No. 3 (1976), 393-416.

[14] T.W. Kibble, Lorentz invariance and the gravitational field, *J. Math. Phys.*, **2** (1961), 212.

[15] J.W. Moffat, Geometrical gauge theory of gravity and elementary particle forces, *Phys. Rev. D*, **17**, No. 3 (1978), 1465-1971.