International Journal of Pure and Applied Mathematics
Volume 49 No. 4 2008, 591-598

CALCULATING ISOMETRY GROUPS IN GENERAL RELATIVITY

José Arnaldo F. Roveda[1 §], James E.F. Skea[2], Marcelo E. Araújo[3]

[1]UNESP - Sao Paulo State University
Sorocaba Campus, Sorocaba, 18087-180, BRAZIL
e-mail: roveda@sorocaba.unesp.br

[2]UERJ - Rio de Janeiro State University
Rio de Janeiro, BRAZIL
e-mail: jimsk@dft.if.uerj.br

[3]UFRJ - Federal University of Rio de Janeiro
Rio de Janeiro, BRAZIL
e-mail: marcelo@if.ufrj.br

Abstract: This paper treats the equivalence problem for General Relativity (GR), which decides whether or not two metrics describe the same space-time. Part of this problem involves calculating invariant characteristics of the space-time. One of the characteristics currently calculated by programs that implement the equivalence problem is the dimension of the isometry group. It is more useful to know the actual isometry group. Here we present the package *isometry*, implemented in the computer algebra system *Maple* which, given a basis for an orbifold, calculates the isometry group if this is of dimension 2, 3 or 4. An example of the use of the package is shown.

AMS Subject Classification: 78A88
Key Words: general relativity, calculating invariant characteristics of the space-time, *Maple*

1. Introduction

The equivalence problem in GR [3], consists of determining whether or not two metrics locally represent the same space-time. One approach is to calculate the components of spinors related to the curvature in a canonical basis. Comparing

Received: August 14, 2008

§Correspondence author

components from two classifications we obtain a system of algebraic equations. If there is a consistent solution to the equations, the metrics locally represent the same space-time.

For metrics which locally represent different space-times, it may not be necessary to solve any equations to prove inconsistence. All that is needed is to find some invariant information calculated from the metrics which is different: for example the Petrov type of the Weyl spinor, the isotropy group, or the Segre type of the Ricci spinor.

An invariant calculated in programs [7] which implement the invariant classification is the dimension of the isometry group. At times this information is not sufficiently precise to distinguish different space-times. For example, space-times of classes Bianchi VIII and Bianchi IX [4] have isometry groups of dimension 3, but are distinct. When the isometry group is of dimension 2, 3 or 4 we seek to refine the invariant information available by calculating the isometry group in question.

This work is a development of Araújo et al [1], who showed how to obtain a basis for the manifold of the isometry group orbits, and how to calculate by hand some groups. However no general method for how to calculate the isometry group was presented. Here we present the *Maple* package *isometry*, based on [4] and [5] which, given a basis for an orbifold, calculates the isometry group.

In Section 2 we explain the theory behind the classification of the isometry groups; in Section 3 we present the *isometry* package, explain how to use it, and give an example; Section 4 contains the conclusions.

The source code for this package can be found in www.sorocaba.unesp.br/professor/roveda.

2. Isometry Groups in GR

Suppose we have a basis for the manifold of group orbits (orbifold). From this we can calculate the structure constants C^a_{bd} of the isometry group. The structure constants will, in general, not be in their canonical form, making it difficult to identify the isometry group. The main idea behind our package is to transform the basis for the orbifold into one for which the structure constants are in standard form, making it easy to determine the isometry group and the Killing vectors.

We restrict ourselves to isometry groups with dimension, r, equal to 2, 3 and

4. There are few space-times with isometry groups which have $r > 5$ and they are easily recognised. Details about these space-times can be found in [4]. There is at present, no classification of the isometry groups with dimension equal to 5. For isometry groups of order $r < 5$, we have the following panorama:

r = 4: there are 31 groups classified by MacCallum;
r = 3: the 9 Bianchi groups;
r = 2: there are only two groups (Abelian and non-Abelian);
r = 1: trivial.

When $r = 2$ (G_2) the classification is simple: the group is Abelian if $C^a_{\ bd} \equiv 0$, and we write G_2I, and non-Abelian (G_2II) otherwise.

2.1. Isometry Groups of Dimension 3

The classification of these groups begins by calculating the vector $A_b \equiv \frac{1}{2}C^a_{\ ba}$. The vector A_b divides the groups into two classes: G_3A, if $A_b \equiv 0$, and G_3B otherwise.

We then define the symmetric matrix

$$N = N^{de} \equiv \frac{1}{2}C^d_{\ bc}\varepsilon^{bce} - \varepsilon^{def}A_f,$$

with ε^{abc} the completely antisymmetric pseudotensor with $\varepsilon^{123} = 1$ which, in a suitable basis, can be diagonalised such that $N^{de} = \mathrm{diag}(n_1, n_2, n_3)$ with $n_i = 0, \pm 1$, and $i = 1, 2, 3$.

With N diagonalised, the classification is completed by swapping the basis elements so that the n_i coincide with the values in Table 1. Note that the Bianchi VI and VII subgroups can be subdivided according to the value of the constant h defined by $(1-h)C^a_{\ ba}C^d_{\ cd} = -2hC^a_{\ db}C^d_{\ ac}$.

Class	G_3A						G_3B				
Type	I	II	VI_0	VII_0	$VIII$	IX	V	IV	III	VI_h	VII_h
Rank(N)	0	1	2	2	3	3	0	1	2	2	2
$\|\mathrm{sig}(N)\|$	0	1	0	2	1	3	0	1	0	0	2
A	0	0	0	0	0	0	1	1	1	$\sqrt{-h}$	$\sqrt{h}$
n_1	0	1	0	0	-1	1	0	0	0	0	0
n_2	0	0	-1	1	1	1	0	0	-1	-1	1
n_3	0	0	1	1	1	1	0	1	1	1	1

Table 1: Classification of the Bianchi groups

2.2. Isometry Groups of Dimension 4

The treatment here follows that of MacCallum [5] and we refer to that work for a full treatment.

From the structure constants we form the same vector $A_b \equiv C^a_{\ ab}$ and divide the groups into two classes according to whether $A_b = 0$ (Unimodular) or $A_b \neq 0$ (Non-unimodular).

Now the structure constants $C^a_{\ bd}$ satisfy:

$$C^a_{\ bd} = -C^a_{\ db} \text{ and } C^e_{\ [bc}C^a_{\ d]e} = 0. \tag{1}$$

In the non-unimodular case it is possible to choose a basis in which

$$A_b = (0, 0, 0, A). \tag{2}$$

Using (1) it can be shown that

$$A_b C^b_{\ ad} = 0. \tag{3}$$

From (2) and (3) we find that $C^4_{\ ab} = 0$, for $a, b = 1, \ldots 4$. With this in mind, we need only study the components $C^\alpha_{\ \beta\gamma}$ and $C^\alpha_{\ \beta 4} \equiv \theta^\alpha_{\ \beta}$, with $\alpha, \beta, \gamma = 1, 2$ and 3.

Now define the symmetric matrix $n^{ab} = C^a_{\ de}\varepsilon^{debf}A_f/2A$, where ε^{abcd} is the totally antisymmetric tensor with $\varepsilon^{1234} = 1$. It follows that $n^{a4} = 0$, for $a = 1, \ldots 4$. We then obtain a basis such that $n^{\alpha\beta} = \text{diag}(n_1, n_2, n_3)$ with $n_i = 0$ or ± 1 and α, $\beta = 1$, 2 e 3. It is shown in [5] that the rank of $n^{ab} \leq 2$.

The classification of the non-unimodular groups is based on the rank and modulus of the signature of n^{ab} and the Segre type of the matrix $\theta^a_{\ b}$. In what follows we use the notation of [5].

The non-unimodular groups are represented by the letter N followed by the rank of n^{ab}. This class splits into 3 subclasses, $N2$, $N1$ and $N0$. The subclass $N2$ is followed by the modulus of the signature of n^{ab}, while $N1$ and $N0$ are followed by the Segre type of the matrix $\theta^\alpha_{\ \beta}$, with the following notation: each number in square brackets represents an eigenvector and its multiplicity ([1,1,1],[2,1],[3],...); if two eigenvectors have the same eigenvalue, they appear in parentheses ([(1,1)1],[(1,1,1)],[(2,1)],...); when the eigenvalues are complex, we use the notation z, $\bar{z}$; finally if the eigenvalues have particular values, these appear after the square brackets ($[1,1,1]_{1,\lambda,0}$, $[(1,1)1]_{0,0,1}$, ...).

For the unimodular algebras we have the theorem

Theorem 1. (see [2]) *For a unimodular algebra of dimension 4, there exist*

p_a such that

$$C^a_{\ bd} = \theta^a_{\ [b} p_{d]} \ \ , \ \ \theta^a_{\ b} p_a = 0, \tag{4}$$

or non-zero l^c such that

$$C^a_{\ bd} l^d = 0. \tag{5}$$

This divides the unimodular algebras into two subclasses: U1, which satisfy (4) and U3, satisfying (5). The U1 subclass is classified according to the Segre type of θ^α_β. To classify U3, we first find a basis in which $l^c = (0, 0, 0, 1)$, then calculate the matrix $n^{\alpha\beta}$ given by

$$n^{\alpha\beta} = \varepsilon^{\alpha\gamma\delta} C^\beta_{\gamma\delta}/2, \tag{6}$$

and again examine the modulus and signature of $n^{\alpha\beta}$ to terminate the classification.

Table 2 shows the number and types of isometry groups with dimensions 2, 3 and 4.

Dimension	Number	
2	2	Abelian : G_2I Non-Abelian: G_2II
3	11	G_3A: Bianchi I, II, VI_0 VII_0, $VIII$, IX G_3B: Bianchi V, IV III, VI_h, VII_h
4	31	Unimodular: $U1[1,1,1]$, $U1[(1,1,1)]$, $U1[(1,1)1]$, $U1[1,1,1]_{1,-1,0}$, $U1[z,\bar{z},1]$, $U1[2,1]$, $U1[(2,1)]$, $U1[3]$, $U3I0$, $U3I2$, $U3S1$, $U3S3$ Non-Unimodular: $N22$, $N20$, $N1[1,1]$, $N1[(1,1)]$,$N1[1,1]_{2,0}$, $N1[z,\bar{z}]$, $N1[2]$, $N0[1,1,1]$, $N0[(1,1)1]$, $N0[(1,1,1)]$, $N0[(1,1)1]_{1,-1,0}$, $N0[1,1,1]_{1,\mu,0}$ $N0[(1,1)1]_{0,0,1}$, $N0[z,\bar{z},1]$, $N0[2,1]$, $N0[(2,1)](1)0$, $N0[(2,1)](2)0$, $N0[3]$

Table 2: Isometry groups of orders 2, 3 and 4

3. Using the *Isometry* Package

The *isometry* package is loaded with the command

```
> read("isometry.map");
```

where the file isometry.map contains the source code. The user then supplies two pieces of data:

- **V**, a list of coordinates for the orbifold;
- **B1**, a list of lists, containing the components of the basis 1-forms on the orbifold.

Having defined these objects the user types

```
> isometry(V,B1);
```

and the calculations necessary to obtain the group in question and the associated Killing vectors will be performed by the program.

The package also allows for the user to supply a set of vectors for the manifold of the group orbits. In this case the user should define a variable (lists of lists) **B** containing this basis and type

```
> isometry(V,vectors = B);
```

3.1. Example

This metric, discovered independently by McLenaghan and Tariq [6] and Tupper [9], represents a homogeneous space-time with an electromagnetic field and can be written as

$$\mathrm{d}s^2 = \frac{a^2}{x^2}(\mathrm{d}x^2 + \mathrm{d}y^2) + x^2\mathrm{d}p^2 - (\mathrm{d}t - 2y\mathrm{d}p)^2.$$

The metric is of Petrov type I and so has no isotropy group. The space-time is homogeneous and so the isometry group has dimension 4. We can use the original metric as the line element for the group orbifold.

The input data are:

```
> V:=[t,p,x,y];
```

$$V := [t,\, p,\, x,\, y]$$

```
> B1:=[[1,-2*y,0,0],[0,x,0,0],[0,0,a/x,0], [0,0,0,a/x]];
```

$$B1 := [[1,\, -2\,y,\, 0,\, 0],\, [0,\, x,\, 0,\, 0],\, [0,\, 0,\, \frac{a}{x},\, 0],\, [0,\, 0,\, 0,\, \frac{a}{x}]]$$

To calculate a vector basis we use:

```
> inverse(B1):
```

We can now supply this vector basis as input and obtain:

```
> isometry(V,vectors=[[1,2*y/x,0,0], [0,1/x,0,0],
      [0,0,x/a,0],[0,0,0,x/a]]);
```

"U3I0"

3.2. Conclusions

The objective of this work was to develop a package that calculates the isometry group for a manifold of group orbits, with the intention that the package can help in future research in General Relativity and, in particular, the Equivalence Problem. The *isometry* package was tested with examples of groups of dimension 2, 3 e 4, with satisfactory results. Within these results we would highlight the classification of the Bianchi groups, which coincides with those given in [4]. For groups of dimension 4, we believe that this is the first classification for the space-times presented here (at least using MacCallum's classification scheme [5]). The intention is that the package will be integrated with the invariant classification package in GRTensor [8], which would provide a basis for the orbits from the isotropy group of the space-time and the functionally independent functions of the coordinates which appear in the invariant classification. It would be interesting in the future to extend the classification to isometry groups of dimension 5.

References

[1] M.E. Araújo, T. Dray, J.E.F. Skea, *Gen. Rel. Grav.*, **24** (1992), 477.

[2] D.L. Farnsworth, R.P. Kerr, *Gen. Rel. Grav.*, **12** (1966), 693.

[3] A. Karlhede, *Gen. Rel. Grav.*, **12** (1980), 693.

[4] D. Kramer, H. Stephani, M.A.H. MacCallum, E. Herlt, *Exact Solution of Einstein's Field Equations*, Cambridge University Press, Cambridge (1980).

[5] M.A.H. MacCallum, *On the Classification of the Real Four-Dimensional Lie Algebras*, On Einstein's Path-Essays in Honor of Engelbert Schucking, Ed. Alex Harvey, Spring-Verlag, New York (1999), 299.

[6] R.G. McLenaghan, N. Tariq, *J. Math. Phys.*, **16** (1975).

[7] M.B. Monaghan, K.O. Geddes, K.M. Heal, G. Labahn, S.M. Vorkoeter, J. MaCarron, P. DeMarco, *Maple 7 Programming Guide*, Waterloo Maple Inc (2001).

[8] P. Musgrave, D. Pollney, K. Lake, GRTensor II: A package for general relativity, *Fields Institute Comm.*, **15** (1996), 313.

[9] B.O.J. Tupper, *Gen. Rel. Grav.*, **7** (1976), 479.