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MODIFIED HOMOTOPY PERTURBATION METHOD FOR SOLVING LINEAR AND NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract: In this paper, the modified homotopy perturbation method is adopted for solving linear and nonlinear Schrödinger equations and the solutions of some Schrödinger equations are exactly obtained in the form of convergent Taylor series. The results show that this method is very efficient and convenient and can be applied to a large class of problems. Some examples are tested to show the pertinent features of this method.

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Key Words: modified homotopy perturbation method (MHPM), Schrödinger equation

1. Introduction

The Schrödinger equation is the basic of quantum mechanics, see [1], [5]. This model equation also arises in many other practical domains of physical and technological interest, e. g. optics, seismology and plasma physics. There are a lot of studies on the numerical solution of initial and initial boundary problems for solving the linear and nonlinear Schrödinger equation, see [2], [3].

The homotopy perturbation method (HPM) is a new approach which searches for an analytical approximate solution of linear and nonlinear problems. The HPM has been applied to Volterra's integro differential equation [4], to nonlinear oscillators [7], nonlinear wave equations [10] and boundary value problems [11], and to other fields [6], [8], [9]. In [12], Odibat and Momani modified the

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HPM to solve nonlinear differential equations of fractional order. This modification reduces the nonlinear fractional differential equations to a set of linear ordinary differential equations. In this paper we apply MHPM to obtain the exact solutions for linear and nonlinear Schrödinger equations.

The organization of this paper is as follows. In Section 2, we introduce the model of the problem. In Section 3, we apply MHPM in a direct manner to establish exact solutions for linear and nonlinear Schrödinger equations. In Section 4, we describe the numerical solution of linear and nonlinear Schrödinger equations to show the power of the method in a unified manner without requiring any additional restriction.

2. The Model of the Problems

In this paper, the linear Schrödinger equation is considered as follows:

$$\frac{\partial\psi}{\partial t}+i\frac{\partial^2\psi}{\partial x^2}=0,\qquad \psi(x,0)=g(x),\qquad i^2=-1, \tag{1}$$

and we consider the nonlinear Schrödinger equation of the form:

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+\beta|\psi|^2\psi=0,\qquad \psi(x,0)=g(x),\qquad i^2=-1, \tag{2}$$

where $|\psi|^2=\psi\overline{\psi}$, β is a real constant and $\psi(x,t)$ is a complex function.

3. MHPM Method

The HPM, which provides an analytical approximate solution, is applied to various nonlinear problems [4], [7]. In this section, we recapitulate a modification of the HPM introduced in [12]. To illustrate the basic ideas of the modification, we consider the following nonlinear differential equation:

$$L(\Psi)+N(\Psi)=f(r),\quad r\in\Omega, \tag{3}$$

with the initial conditions:

$$\Psi_j(x,0)=g_j(x),\quad j=0,1,2,..., \tag{4}$$

where L is a linear operator, N is a nonlinear operator, f is a known analytic function.

In view of the homotopy technique, we can construct the following homotopy:

$$\frac{\partial\psi}{\partial t} + L(\psi) - f(r) = p[\frac{\partial\psi}{\partial t} - N(\psi)], \quad p \in [0,1], \tag{5}$$

or

$$\frac{\partial\psi}{\partial t} - f(r) = p[\frac{\partial\psi}{\partial t} - L(\psi) - N(\psi)], \quad p \in [0,1]. \tag{6}$$

The homotopy parameter p always changes from zero to unity. In case $p = 0$, equation (5) becomes the linearized equation:

$$\frac{\partial\psi}{\partial t} + L(\psi) = f(r),$$

and equation (6) becomes the linearized equation:

$$\frac{\partial\psi}{\partial t} = f(r).$$

In case $p = 1$, equations (5) or (6) turns out to be the original differential equation (3). The basic assumption is that the solution of equations (5) or (6) can be written as a power series in p :

$$\psi = \psi_0 + p\psi_1 + p^2\psi_2 + p^3\psi_3 + \cdots . \tag{7}$$

Substituting (7) into equations (5) or (6), and equating the terms with identical powers of p, we can obtain a series of linear equations of the form as follows:

$$\begin{aligned}
p^0 &: \frac{\partial\psi_0}{\partial t} + L_0(\psi_0) = f(r), \quad \psi_0(x,0) = g_0(x),\\
p^1 &: \frac{\partial\psi_1}{\partial t} + L_1(\psi_0,\psi_1) = \frac{\partial\psi_0}{\partial t} - N_0(\psi_0), \quad \psi_1(x,0) = 0,\\
p^2 &: \frac{\partial\psi_2}{\partial t} + L_2(\psi_0,\psi_1,\psi_2) = \frac{\partial\psi_1}{\partial t} - N_1(\psi_0,\psi_1), \quad \psi_2(x,0) = 0,\\
p^3 &: \frac{\partial\psi_3}{\partial t} + L_3(\psi_0,\psi_1,\psi_2,\psi_3) = \frac{\partial\psi_2}{\partial t} - N_2(\psi_0,\psi_1,\psi_2), \quad \psi_3(x,0) = 0,\\
&\vdots
\end{aligned} \tag{8}$$

or the form:

$$\begin{aligned}
p^0 &: \frac{\partial\psi_0}{\partial t} = f(r), \quad \psi_0(x,0) = g_0(x),\\
p^1 &: \frac{\partial\psi_1}{\partial t} = \frac{\partial\psi_0}{\partial t} - L_0(\psi_0) - N_0(\psi_0), \quad \psi_1(x,0) = 0,\\
p^2 &: \frac{\partial\psi_2}{\partial t} = \frac{\partial\psi_1}{\partial t} - L_1(\psi_0,\psi_1) - N_1(\psi_0,\psi_1), \quad \psi_2(x,0) = 0,\\
p^3 &: \frac{\partial\psi_3}{\partial t} = \frac{\partial\psi_2}{\partial t} - L_2(\psi_0,\psi_1,\psi_2) - N_2(\psi_0,\psi_1,\psi_2), \quad \psi_3(x,0) = 0,\\
&\vdots
\end{aligned} \tag{9}$$

respectively, where the terms $L_0, L_1, L_2, ...$ and $N_0, N_1, N_2, ...$ satisfy the following equations:

$$L(\psi_0 + p\psi_1 + p^2\psi_3 + ...) = L_0(\psi_0) + pL_1(\psi_0, \psi_1) + p^2 L_2(\psi_0, \psi_1, \psi_2) + ...,$$

$$N(\psi_0 + p\psi_1 + p^2\psi_3 + ...) = N_0(\psi_0) + pN_1(\psi_0, \psi_1) + p^2 N_2(\psi_0, \psi_1, \psi_2) + \cdots .$$

Setting $p = 1$ in (7), results in the approximate solution of (3):

$$\Psi = \lim_{p\to 1} \psi = \psi_0 + \psi_1 + \psi_3 + \cdots .$$

It obvious that the linear equations in (8) or (9) are easy to solve, and the components ψ_n, $n \geq 0$ of the HPM can be completely determined, and the series solutions are thus entirely determined.

4. Numerical Examples

Example 4.1. We consider the linear Schrödinger equation as follows:

$$\begin{aligned} \frac{\partial \psi}{\partial t} &+ i\frac{\partial^2 \psi}{\partial x^2} = 0, \qquad (10) \\ \psi(x,0) &= e^{2ix+1}. \end{aligned}$$

According to the homotopy (9), we obtain the following set of linear partial differential equations:

$$p^0 : \frac{\partial \psi_0}{\partial t} = 0, \quad \psi(x,0) = e^{2ix+1}, \qquad (11)$$

$$p^1 : \frac{\partial \psi_1}{\partial t} = -i\frac{\partial^2 \psi_0}{\partial x^2}, \quad \psi_1(x,0) = 0, \qquad (12)$$

$$p^2 : \frac{\partial \psi_2}{\partial t} = -i\frac{\partial^2 \psi_1}{\partial x^2}, \quad \psi_2(x,0) = 0, \qquad (13)$$

$$p^3 : \frac{\partial \psi_3}{\partial t} = -i\frac{\partial^2 \psi_2}{\partial x^2}, \quad \psi_3(x,0) = 0. \qquad (14)$$

Solving equations (11)-(14), we obtain:

$$\begin{aligned} \psi_0(x,t) &= e^{2ix+1}, \\ \psi_1(x,t) &= 4ite^{2ix+1}, \\ \psi_2(x,t) &= -8t^2 e^{2ix+1}, \\ \psi_3(x,t) &= -\frac{32}{3} it^3 e^{2ix+1}. \end{aligned}$$

The solution of equation (10), when $p \to 1$, will be as follows:

$$\psi(x,t) = e^{2ix+1}(1 + (4it) + \frac{(4it)^2}{2!} + \frac{(4it)^3}{3!} + ...).$$

We have exact solution as follows:

$$\psi(x,t) = e^{2i(x+2t)+1}.$$

Example 4.2. We consider the linear Schrödinger equation as follows:

$$\begin{aligned} \frac{\partial \psi}{\partial t} &+ i\frac{\partial^2 \psi}{\partial x^2} = 0, \\ \psi(x,0) &= \sinh(3x-1). \end{aligned} \tag{15}$$

According to the homotopy (9), we obtain the following set of linear partial differential equations:

$$p^0 \; : \; \frac{\partial \psi_0}{\partial t} = 0, \quad \psi(x,0) = \sinh(3x-1), \tag{16}$$

$$p^1 \; : \; \frac{\partial \psi_1}{\partial t} = -i\frac{\partial^2 \psi_0}{\partial x^2}, \quad \psi_1(x,0) = 0, \tag{17}$$

$$p^2 \; : \; \frac{\partial \psi_2}{\partial t} = -i\frac{\partial^2 \psi_1}{\partial x^2}, \quad \psi_2(x,0) = 0, \tag{18}$$

$$p^3 \; : \; \frac{\partial \psi_3}{\partial t} = -i\frac{\partial^2 \psi_2}{\partial x^2}, \quad \psi_3(x,0) = 0. \tag{19}$$

Solving equations (16)-(19), we obtain:

$$\begin{aligned} \psi_0(x,t) &= \sinh(3x-1), \\ \psi_1(x,t) &= -i9t\sinh(3x-1), \\ \psi_2(x,t) &= -\frac{81}{2}t^2\sinh(3x-1), \\ \psi_3(x,t) &= -i\frac{243}{2}t^3\sinh(3x-1)\,. \end{aligned}$$

The solution of equation (15), when $p \to 1$, will be as follows:

$$\psi(x,t) = \sinh(3x-1)(1 - 9it + \frac{(-9it)^2}{2!} + \frac{(-9it)^3}{3!} + ...)\,.$$

We have exact solution as follows:

$$\psi(x,t) = e^{-9it}\sinh(3x-1).$$

Example 4.3. We consider the nonlinear Schrödinger equation as follows:

$$\begin{aligned} i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} - 2|\psi|^2\psi &= 0, \\ \psi(x,0) &= e^{ix+1}. \end{aligned} \tag{20}$$

According to the homotopy (9), we obtain the following set of linear partial differential equations:

$$p^0 \; : \; \frac{\partial \psi_0}{\partial t} = 0, \quad \psi(x,0) = e^{ix+1}, \tag{21}$$

$$p^1 \quad : \quad \frac{\partial \psi_1}{\partial t} = -i\frac{\partial^2 \psi_0}{\partial x^2} + 2\psi_0^2\overline{\psi_0}, \qquad \psi_1(x,0) = 0, \tag{22}$$

$$p^2 \quad : \quad i\frac{\partial \psi_2}{\partial t} = -\frac{\partial^2 \psi_1}{\partial x^2} + 2\psi_0^2\overline{\psi_1} + 4\psi_0\psi_1\overline{\psi_0}, \qquad \psi_2(x,0) = 0, \tag{23}$$

$$\begin{aligned} p^3 \quad &: \quad i\frac{\partial \psi_3}{\partial t} = -\frac{\partial^2 \psi_2}{\partial x^2} + 2\psi_0^2\overline{\psi_2} + 4\psi_0\psi_1\overline{\psi_1} + 2\psi_1^2\overline{\psi_0} + 4\psi_0\psi_2\overline{\psi_0}, \\ \psi_3(x,0) \quad &= \quad 0. \end{aligned} \tag{24}$$

Solving equations (21)-(24), we obtain:

$$\begin{aligned} \psi_0(x,t) &= e^{ix+1}, \\ \psi_1(x,t) &= -3ite^{ix+1}, \\ \psi_2(x,t) &= -\frac{9}{2}t^2e^{ix+1}, \\ \psi_3(x,t) &= \frac{9}{2}it^3e^{ix+1}. \end{aligned}$$

The solution of equation (20), when $p \to 1$, will be as follows:

$$\psi(x,t) = e^{ix+1}(1 - 3it + \frac{(-3it)^2}{2!} + \frac{(-3it)^3}{3!} + ...)\,.$$

We have exact solution as follows:

$$\psi(x,t) = e^{i(x-3t)+1}.$$

5. Conclusions

In this paper, we apply modified homotopy perturbation method for solving linear and nonlinear Schrödinger equations. This method is powerful and efficient technique for finding exact as well as approximate solutions for wide classes of linear and nonlinear Schrödinger equations.

References

[1] S. Brandt, H.D. Dahmen, *The Picture Book of Quantum Mechanics*, Springer, Berlin (1995).

[2] T.F. Chan, D. Lee, L. Shen, Stable explicit schemes for equations of the Schrödinger type, *SIAM J. Numer. Anal.*, **23** (1986), 274-281.

[3] J.B. Chen, M.Z. Qin, Multi-symplectic Fourier pseuospectral method for

the nonlinear Schrödinger equation, *Electronic Trans Numer. Anal.*, **12** (2001), 193-204.

[4] M. El-Shahed, Application of He's homotopy perturbation method to Volterra's integro-differential equation, *Int. J. Nonlinear Sci. Numer. Simul.*, **6**, No. 2 (2005), 163-168.

[5] D.J. Griffiths, *Introduction to Quantum Mechanics*, Prentice-Hall, Englewood Cliffs, NJ (1995).

[6] J.H. He, Homotopy perturbation method: A new nonlinear analytic technique, *Appl. Math. Comput.*, **135** (2003), 73-79.

[7] J.H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, *Appl. Math. Comput.*, **151** (2004), 287-292.

[8] J.H. He, Asymptotology by homotopy perturbation method, *Appl. Math. Comput.*, **156** (2004), 591-596.

[9] J.H. He, Limit cycle and bifuraction of nonlinear problems, *Chaos Solitons Fractals*, **26**, No. 3 (2005), 827-833.

[10] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, *Chaos Solitons Fractals*, **26**, No. 3 (2005), 695-700.

[11] J.H. He, Homotopy perturbation method for solving boundary value problems, *Phys. Lett. A*, **350**, No-s: 1-2 (2006), 87-88.

[12] Z. Odibat, S. Momani, Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order, *Chaos Solitons Fractals* (2006).