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EXISTENCE/NON-EXISTENCE OF CONFORMAL MAPPING IN SOME FAMILY OF FUNCTIONS

Yoshihiro Mochimaru
Department of International Development Engineering
Graduate School of Science and Engineering
Tokyo Institute of Technology
Meguro-ku, Tokyo 152−8550, JAPAN
e-mail: ymochima@o.cc.titech.ac.jp

Abstract: Existence/non-existence of conformal mapping for a family of functions is investigated to show breaking down range of a parameter concretely.

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1. Introduction

Techniques of conformal mapping cover many classical applications in mathematical physics. For such purposes one-to-one correspondence in mapping is srongly requested, although it is not necessarily automatically satisfied for any given specified mapping function. Such cases are reported, e.g. by Hartmann and Opfer [1], and Mochimaru [2].

2. Analysis

2.1. Genaral

Consider the following conformal mapping from the closed domain $D + \partial D$ in

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z to w:

$$w = F(z),$$

where F is a single valued analytic function. If one-to-one correspondence is assumed, then F^{-1} is a single valued function, the necessary condition of which is

$$F'(z) \neq 0, \quad z \in D.$$

2.2. Mapping from a Nearly Eccentric Annular to a Rectangular Domain

Consider the following mapping from z to w

$$-i \sinh \alpha_0 \coth(w/2) = z + \epsilon \sin(nz),$$

$$0 < \alpha_0 \leq \Re(w) \leq \alpha_1, \quad -\pi < \Im(w) \leq \pi, \quad n > 0, \quad z \equiv x + iy. \tag{1}$$

The case $\epsilon = 0$ gives a mapping from an exactly eccentric annular domain to a rectangular area. We restrict ourselves to the domain z such that $|z + i \cosh \alpha_0| \leq 1$ and $|z + i \sinh \alpha_0 \coth \alpha_1| \geq \sinh \alpha_0 / \sinh \alpha_1$ as $|\epsilon| \to 0$. The sufficient condition to get a conformal mapping uniquely for small $|\epsilon|$ is

$$|z + \epsilon \sin nz - c| \geq 1, \tag{2}$$

$$c = -i \cosh \alpha_0 \tag{3}$$

for all possible zeros z of $1 + n\epsilon \cos nz = 0$. The critical zero z for $\epsilon \neq 0$ is given by

$$z = \frac{1}{ni} \ln \left\{ \frac{-1}{\epsilon n} + i \sqrt{1 - \frac{1}{(\epsilon n)^2}} \right\}. \tag{4}$$

Hereafter $\sqrt{\ }$ and ln() stand for a principal value. Figure 1 shows a global exsistence domain inside the configuration for conformal mapping (the sufficient condition, inclusive of $\epsilon = 0$) for a complex parameter ϵ at $\alpha_0 = 0.1, \alpha_1 = 1, n = 5$. Figure 2 shows one case of iso-lines of $\Re(w)$ or $\Im(w) =$ constant for non-existence of conformal mapping (the direction of splitting may be arbitrary), where $\epsilon = 0.01$.

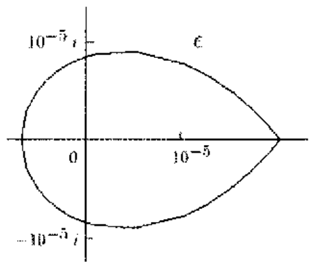

Figure 1: Global existence domain

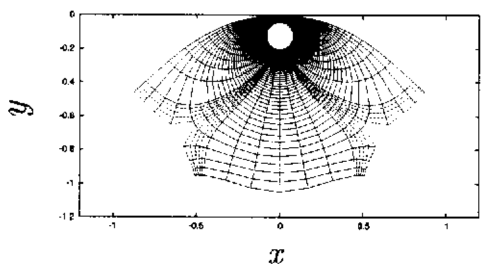

Figure 2: Iso-lines at non-existence case

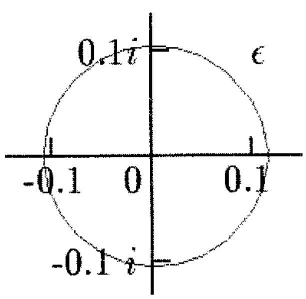

Figure 3: Existence domain at $n = 4$

2.3. Mapping from a Nearly Circular Domain z to a Rectangular Domain w

Consider the following mapping

$$i\mathrm{sn}(w,k) = \frac{z - 1 + \epsilon z^n}{1 + z} \equiv f(z) \quad (\ n : \text{integer} \geq 2\ ), \tag{5}$$

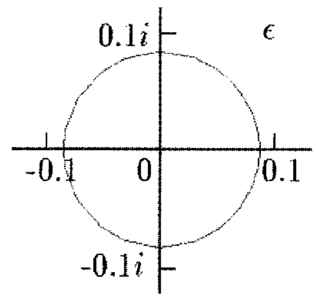

Figure 4: Existence domain at $n = 5$

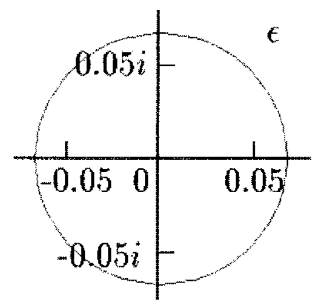

Figure 5: Existence domain at $n = 6$

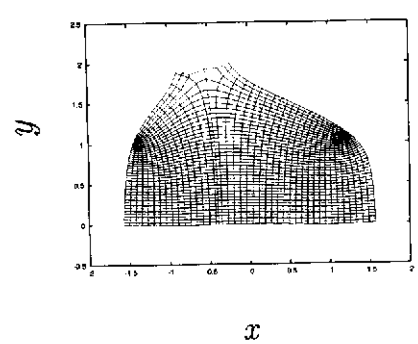

Figure 6: Iso-lines, $\epsilon = 0.01 + 0.015i$

$$-K \leq \Re(w) \leq K, 0 \leq \Im(w) \leq K', \tag{6}$$

where k is a modulus $\in (0, 1)$, K a complete elliptic integral of the first kind ($\equiv K(k)$), and $K' \equiv K(\sqrt{1-k^2})$. In case of $\epsilon = 0$, $|z| \leq 1$. In the said region

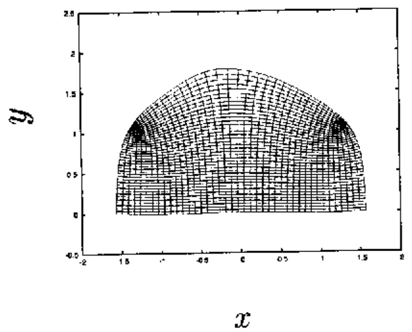

Figure 7: Iso-lines, $\epsilon = 0.002 + 0.015i$

of w, where $|\Re(w)| < K, 0 < \Im(w) < K'$,

$$0 < \Im\{\mathrm{sn}(w,k)\} < +\infty.$$

All the more along the circumference

$$\Im\{\mathrm{sn}(w,k)\} = 0.$$

We restrict ourselves to the domain z such that $|z| \leq 1$ as $|\epsilon| \to 0$. Thus the sufficient condition to get a one-to-one conformal mapping for small $|\epsilon|$ is

$$\Re\{f(z)\} \geq 0 \qquad \text{for all possible zeros } z \text{ of} \quad f'(z) = 0, \quad \text{i.e.,}$$

$$2 + n\epsilon z^{n-1} + (n-1)\,\epsilon z^n = 0. \tag{7}$$

The critical zero z for $n = 3, \epsilon \neq 0$ is given by

$$z = \frac{1}{\sqrt{-2\epsilon}\sin\phi}, \tag{8}$$

$$\phi = \frac{1}{3i}\ln\left(\sqrt{1+\frac{2}{\epsilon}} - \frac{2i}{\sqrt{-2\epsilon}}\right). \tag{9}$$

The critical zero z for $n = 4, \epsilon \neq 0$ is given by

$$\frac{1}{z} = \sqrt{\beta^2 + \frac{3}{2}\beta} + \sqrt{-\beta^2 - \frac{3}{2}\beta - \beta\sqrt{4\beta^2 + 6\beta}}, \tag{10}$$

$$\beta = \frac{1}{\sqrt{8/\epsilon}} \Bigg/ \sin\left\{\frac{1}{3i}\ln\left(\sqrt{1-\frac{\epsilon}{2}} - \frac{i}{\sqrt{2/\epsilon}}\right)\right\}. \tag{11}$$

The critical zero z for $n = 5, \epsilon \neq 0$ is given [3], [4] by

$$z = 2 \times 5^{3/4}\beta^{1/4}\sqrt{1-\beta^2}\left(\frac{-2}{5\epsilon}\right)^{(1/4)} \Bigg/ \phi, \tag{12}$$

$$\beta = \tan\left\{\frac{1}{4i}\ln\left(\sqrt{1+\frac{\epsilon}{2}}+\frac{i}{\sqrt{-2/\epsilon}}\right)\right\}, \tag{13}$$

$$\phi = \sqrt{40}Q^{3/8}\left(1+Q-Q^2+Q^3-8Q^5-9Q^6+8Q^7-9Q^8+\cdots\right), \tag{14}$$

$$\begin{aligned} Q \quad &\equiv \quad \left[\beta^2/16+\beta^4/32+(21/1024)\beta^6+(31/2^{11})\beta^8\right. \\ &\quad \left. +\,(6257/2^{19})\beta^{10}+(10293/2^{20})\beta^{12}+\cdots\right]^{(1/5)}. \end{aligned} \tag{15}$$

The critical zero z for $n=6, \epsilon \neq 0$ is given by

$$\frac{\omega}{z} = 1+\frac{1}{6}\psi-\frac{1}{24}\psi^2+\frac{1}{81}\psi^3-\frac{91}{31104}\psi^4+\cdots, \tag{16}$$

$$\psi \equiv \frac{6}{5\omega}, \quad \omega \equiv \left(\frac{-2}{5\epsilon}\right)^{(1/6)}. \tag{17}$$

Figures 3, 4, and 5 show a global existence closed domain for conformal mapping for a complex parameter ϵ (inclusive of $\epsilon = 0$) at $n = 4, 5$, and 6, respectively.

2.4. Mapping from a Nearly Rectangular Domain z to a Rectangular Domain w

Consider the following mapping

$$\begin{aligned} \mathrm{sn}\left(K'i-\frac{2}{\pi}Kw,k\right) \quad &= \quad \mathrm{sn}\left(K'i-\frac{2}{\pi}Kz,k\right) \\ &\quad +\epsilon\frac{\pi}{2Kk}\coth\left(\frac{\pi K'}{2K}\right)(\cos 2z+1) \equiv f(z), \end{aligned} \tag{18}$$

$$|\Re(w)| \leq \frac{\pi}{2}, \quad 0 \leq \Im(w) \leq \frac{\pi K'}{2K}, \quad z \equiv x+iy, \tag{19}$$

where k is a modulus $\in (0,1)$. In the said region of w where $|\Re(w)| < \pi/2, 0 < \Im(w) < \pi K'/(2K)$,

$$0 < \Im\{\mathrm{sn}(w,k)\} < +\infty.$$

All the more along the circumference

$$\Im\{\mathrm{sn}(w,k)\} = 0.$$

We restrict ourselves to the domain z such that $|\Re(z)| \leq \pi/2, \quad 0 \leq \Im(z) \leq \pi K'/(2K)$. Thus the condition to get a one-to-one conformal mapping for small $|\epsilon|$ is

$\Im\{f(z)\} \leq 0$ for all possible zeros (in the neighbourhood area) z

$$\text{of} \quad f'(z) = 0 \ . \tag{20}$$

Figures 6 and 7 show cases of iso-lines $\Re(w)$ or $\Im(w)$ = constant for non-existence or existence of conformal mapping at $\frac{\pi K'}{2K} = \frac{3}{2}$, $\epsilon = 0.01 + 0.015i$ (see Figure 6) or $\epsilon = 0.002 + 0.015i$ (see Figure 7), respectively.

3. Conclusion

Existence or non-existence of conformal mapping is investigated for some family of functions. As a result, existence of conformal mapping will break down for relatively small values of a parameter.

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