International Journal of Pure and Applied Mathematics
Volume 49 No. 4 2008, 517-524

ON MINIMUM CONCAVE COST NETWORK FLOW PROBLEMS

Dalila B.M.M. Fontes
LIAAD-INESC Porto L.A.
Faculdade de Economia
Universidade do Porto
Rua Dr. Roberto Frias, Porto, 4200-464, PORTUGAL
e-mail: fontes@fep.up.pt

Abstract: Minimum concave Cost Network Flow Problems (MCNFPs) arise naturally in many practical applications such as communication, transportation, distribution, and manufacturing, due to economic considerations. In addition, it has been shown that every MCNFP with general nonlinear cost functions can be transformed into a concave MCNFP on an expanded network. It must also be noted, that multiple source and capacitated networks can be transformed into single source and uncapacitated networks. The main feature defining the complexity of MCNFPs is the type of cost function for each arc. Concave MCNFPs are known to be NP-hard even for the simplest version (i.e. fixed-charge single source and uncapacitated). The review presented in this work describes several approaches to the design of Single Source Uncapacitated (SSU) flow networks involving concave costs.

AMS Subject Classification: 90C35, 90C27, 90C26
Key Words: network flow problems, combinatorial optimization, concave costs

1. Introduction

Network problems trace their roots to the work of Euler, Kirchhoff and other great classical scientists. Major developments in network theory happened after the publication of the first graph theory book (by D. König) in 1936. In the last decade many new developments were achieved and hundreds of books and

Received: August 14, 2008

papers published. Some introducing new problems, others new algorithms or modifications to existing ones, and some surveying work was also carried out.

In this work, we focus on the Single Source Uncapacitated (SSU) MCNFP, which can be used to address a broader class of MCNFPs. General nonlinear MCNFPs can be transformed into concave MCNFPs on an expanded network [22]. In addition, multiple source and capacitated networks can be transformed into single source and uncapacitated networks [27]. Other problems without a natural network formulation have been solved efficiently after being reformulated as network optimization problems [3, 11]. Another motivation for addressing the MCNFP is its frequent applicability to model practical problems. This can be seen from the wide and diverse application areas in which it has been utilized, see e.g. [13] and the references therein, since there are many practical situations where a product is routed through a network. Concave cost functions in network problems arise in practice due to economic considerations, e.g. economies of scale often lead to a decrease in marginal costs and fixed-charge costs may arise, for example, in toll charges on a highway, landing fees at an airport, considering a new customer or a new supply route.

The complexity of concave MCNFPs arises from minimizing a concave function over a convex feasible region, defined by the network constraints, which implies that a local optimum is not necessarily a global optimum. Furthermore, there is no simple criterion for deciding whether a local minimum is also a global minimum. The main feature defining the complexity of MCNFPs is the type of cost function for each arc. A discussion of other parameters affecting problem complexity can be found in [5]. Although concave MCNFPs are known to be NP-*hard* [14] (even for the simplest version, i.e. fixed-charge single source uncapacitated MCNFPs), they do exhibit some special mathematical properties [13]. Concave MCNFPs have the combinatorial property that if a finite solution exists, then an optimal solution occurs at a node (extreme point) of the corresponding feasible domain [27], defined by the network constraints. SSU concave MCNFPs have a finite solution if and only if there exists a direct path going from the source node to every demand node and if there are no negative cost cycles. Therefore, an extreme flow is a tree rooted at the single source spanning all demand nodes [27]. Thus, the objective becomes to find an optimal tree rooted at the source node that satisfies all customers demand at minimum cost.

2. Overview of Existing Methods

Existing algorithms for concave MCNFPs can be characterized in terms of the type of problems they solve and whether the solution provided is exact (a global optimum) or an approximation (a bound). Although this class of problems is known to be NP-*hard*, there are special cases arising from imposing additional structure for which polynomial-time or even strongly polynomial-time algorithms have been developed. The problem types include restrictions on both the objective function and on the underlying network.

2.1. Specific Concave MCNFPs

For example, a special case arises from considering a small and fixed number of nonlinear arcs (the remaining arcs being linear) for which polynomial-time or even strongly polynomial-time algorithms have been developed, see [15].

The MCNFP with fixed-charge costs is a special class of MCNFPs, for which numerous methods have been proposed. Recent works have been reported by Kim and Hooker [21] that have developed a branch-and-bound method, which combines constraint programming techniques (to reduce the number of branches) with linear programming relaxation (to obtain bounds); by Ortega and Wolsey [23] who develop a branch-and-cut method by extending the cutting planes used for solving uncapacitated lot sizing problems previously developed; and by Kim and Pardalos that give a dynamic slope scaling procedure for fixed-charge [19] and piecewise linear concave costs [20]. They solve linear problems that are recursively updated by using the previous solution. At each iteration the feasible domain is reduced by a contraction rule based on the work of Thakur [26].

2.2. General Concave MCNFPs: Exact Methods

Among exact solution methodologies for general concave MCNFP, Branch-and-Bound (BB) has been the most commonly used, however Dynamic Programming (DP) has also been a successful approach.

Gallo et al [12] developed an efficient BB method to solve SSU concave MCNFPs. The branching is done by adding arcs that extend the current subtree, while the bounding is obtained by linear underestimation. An improvement is given in [14], where the bounding process has been enhanced by projecting the lower bound on the cost of extending the current path. The problems considered have concave routing costs, but no fixed costs and just some vertices are demand vertices. Horst and Thoai [16] use linear underestimation by convex envelopes and rectangular partition to develop an improved BB algorithm. In the problems addressed all vertices are demand vertices, however, only some arcs have concave costs (without a fixed cost component), the remainder being linear. In [8], Fontes et al give a BB method where the bounding is performed by lower bounds derived by state space relaxations [10] of a dynamic programming formulation [9]. The search is further restricted by using the upper bounds developed in [7]. The problems considered have general concave cost functions on all arcs and all vertices are demand vertices. A relevant discussion can be found in [18] and a more recent survey in [8].

Most DP methods developed for MCNFPs are problem specific and take advantage of the network structure to increase efficiency, exceptions can be found in [4, 9]. "Send-and-Split" is a DP approach proposed by Erikson et al [4] to solve SSU concave MCNFPs. No computational results are reported, however, it can be seen that the computational time requirements are exponential in the number of vertices. More recently, Fontes et al [9] have developed a DP approach for SSU general concave MCNFPs. Although, their method does not depend on the type and number of concave arc costs, it is only useful for small to medium size problems as computational requirements grow exponentially with problem size.

Other exact approaches have been developed, but with less success. Pardalos [24] discusses a range of enumerative techniques based on extreme point ranking, while Horst and Tuy [17] discuss Decomposition techniques.

2.3. General Concave MCNFPs: Heuristic Methods

Due to the complexity of nonlinear network optimization problems and to the fact that efficient methods exist only for highly structured subclasses or small size problems, considerable effort in the research community has been devoted to approximate methods.

Burkard et al [2] consider acyclic graphs with small degree vertices and general concave cost. Linear approximations in a DP approach are obtained. For network flow problems defined on complete graphs, they solve small and medium size problems with large time requirements (over 13 hours).

Local search methods take advantage of the knowledge that an optimal solution to the SSU concave MCNFP is an extreme flow. The main advantage is that the search is confined to extreme flows. The main drawback is that there are many solutions, where the method can get trapped, that are local optima but not global optima. An obvious way of trying to avoid such entrapment is to repeat the local search many times, each time with a different initial solution. Such a local search method has been developed by Fontes et al [7]. An important feature of their algorithm is that valuable information is obtained from the structure of a lower bound solution to the concave MCNFP in order to generate initial feasible solutions. The lower bounds are derived from a state space relaxation and improved by a state space ascent procedure [8] based on a DP formulation of the original problem [9].

A tabu search method embedding a local search method previously developed to solve the multicommodity SSU concave MCNFP can be found in [1]. This improves the local search by allowing movements to a worse solution in the hope that a better one can be found if a different search pattern is used. To prevent cycling, a list of solutions visited is kept in order to avoid going back to solutions already visited. The problems addressed consider only some vertices to be demand vertices and concave arc cost functions having no fixed cost.

In [25] Smith and Walters describe a genetic algorithmic approach to find optimal trees on networks at minimum cost. The initial population is made up of randomly generated feasible trees. A probability, which is proportional to the fitness is computed and assigned to each tree. After two trees T_1 and T_2 have been chosen, the two graphs are superimposed to form a new directed graph G. All arcs in trees T_1 and T_2 will be in graph G and will retain their directions. Mutations are introduced by randomly adding extra arcs to the graph G. It is always possible to generate at least two feasible children as the parents are spanning trees and mutation can only add extra arcs. The method is tested on SSU concave MCNFP where all vertices are demand vertices and all arcs have concave routing costs, given by the square root of the flow, and no fixed cost component. Results are provided for small to medium size problems however, the authors do not report on solution quality nor on computational time requirements.

In [6] a Hybrid Genetic Algorithm (HGA), where a local search strategy is incorporated into a genetic algorithm, has been proposed to solve the SSU MCNFP with general concave costs. The Local Search algorithm tries to improve the solutions in the population by using domain-specific information, while the genetic algorithm recombines good solutions in order to investigate different regions of the solution space. The computational results are compared with results reported in literature and these comparisons allow for the conclusion that the HGA is not only effective but also efficient for finding good solutions to the SSU concave MCNFPs. The authors also compare the HGA results with the results obtained by the genetic algorithm alone. This comparison shows the effectiveness of using the hybrid approach.

References

[1] C. Bazlamaçci, K. Hindi, Enhanced adjacent extreme-point search and tabu search for the minimum concave-cost uncapacitated transshipment problem, *J. the Oper. Res. Soc.*, **47** (1996), 1150-1165.

[2] R.E. Burkard, H. Dollani, P.H. Thach, Linear approximations in a dynamic programming approach for the uncapacitated single-source minimum concave cost network flow problem in acyclic networks, *J. Glob. Optim.*, **19** (2001), 121-139.

[3] E. Elia, *Optimal Coorporate Structuring for Multinationals*, Ph.D. Thesis, Imperial College Business School, London (1997).

[4] R.E. Erickson, C.L. Monma, A.F. Veinott Jr. Send-and-split method for min-concave-cost network flows, *Math. Opns. Res.*, **12** (1987), 634-664.

[5] D.B. M.M. Fontes, *Optimal Network Design Using Nonlinear Cost Flows*, Ph.D. Thesis, Imperial College Business School, London (2000).

[6] D.B. M.M. Fontes, J.F. Gonçalves, Heuristic solutions for general concave minimum cost network flow problems, *Networks*, **50** (2007), 67-76.

[7] D.B. M.M. Fontes, E. Hadjiconstantinou, N. Christofides, Upper bounds for single source uncapacitated minimum concave-cost network flow problems, *Networks*, **41** (2003), 221-228.

[8] D.B. M.M. Fontes, E. Hadjiconstantinou, N. Christofides, A branch-and-bound algorithm for concave network flow problems, *J. Glob. Optim.*, **34** (2006), 127-155.

[9] D.B. M.M. Fontes, E. Hadjiconstantinou, N. Christofides, A dynamic programming approach for solving single-source uncapacitated concave minimum cost network flow problems, *Eur. J. Oper. Res.*, **174** (2006), 1205-1219.

[10] D.B. M.M. Fontes, E. Hadjiconstantinou, N. Christofides, Lower bounds from state space relaxations for concave network flow problems, *J. Glob. Optim.*, **34** (2006), 97-125.

[11] S. Forjanis, B.W. Lamar, Cash flow management network models with discounting, *Omega*, **22** (1994), 149-155.

[12] G. Gallo, C. Sandi, C. Sodini, An algorithm for the min concave cost flow problem, *Eur. J. Opns. Res.*, **4** (1980), 249-255.

[13] G.M. Guisewite, Network problems, In R. Horst, P.M. Pardalos, editors, *Handbook of Global Optimization* Kluwer Academic, Dordrecht (1994).

[14] G.M. Guisewite, P.M. Pardalos, Algorithms for the single-source uncapacitated minimum concave-cost network flow problem, *J. Glob. Optim.*, **3** (1991), 245-265.

[15] G.M. Guisewite, P.M. Pardalos, A polynomial time solvable concave network flow problem, *Networks*, **23** (1993), 143-147.

[16] R. Horst, N.V. Thoai, An integer concave minimization approach for the minimum concave cost capacitated flow problem on networks, *OR Spectrum*, **20** (1998), 45-53.

[17] R. Horst, H. Tuy, *Global Optimization and Deterministic Approaches*, Springer-Verlag, 3-rd Edition (1996).

[18] D. Kennedy, Some branch-and-bound techniques for nonlinear optimization, *Math. Progr.*, **42** (1988), 147-169.

[19] D. Kim, P.M. Pardalos, A solution approache to the fixed charge network flow problem using a dynamic slope scaling procedure, *Opns. Res. Letters*, **24** (1999), 195-203.

[20] D. Kim, P.M. Pardalos, Dynamic slope scaling and trust interval techniques for solving concave piecewise linear network flow problems, *Networks*, **35** (2000), 216-222.

[21] H.-J. Kim, J.N. Hooker, Solving fixed-charge network flow problems with a hybrid Optimization and constraint programming approach, *Annals Opns. Res.*, **115** (2002), 95-124.

[22] B.W. Lamar, A method for solving network flow problems with general nonlinear arc costs, In D.-Z. Du, P.M. Pardalos, editors, *Network Optimization Problems*. World Scientific (1993).

[23] F. Ortega, L.A. Wolsey, A branch-and-cut algorithm for the single-commodity, uncapacitated, fixed-charge network flow problem, *Networks*, **41** (2003), 143-158.

[24] P.M. Pardalos, Enumerative techniques for solving some nonconvex Global optimization problems, *OR Spectrum*, **10** (1988), 29-35.

[25] D.K. Smith, G.A. Walters, Evolutionary approach for finding optimal trees in undirected networks, *Eur. J. Oper. Res.*, **120** (2000), 593-602.

[26] L.S. Thakur, Domain contraction in nonlinear programming: Minimizing a quadratic concave objective over a polyhedron, *Math. Opns. Res.*, **16** (1991), 390-407.

[27] W.I. Zangwill, Minimum concave cost flows in certain networks, *Manag. Sci.*, **14** (1968), 429-450.