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COMBINATIONS OF ORTHOGONAL POLYNOMIALS

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Abstract: Let $\{P_n(x)\}_{n=0}^{\infty}$ be an orthogonal sequence of polynomials. We consider two families of polynomials defined by $Q_n(x) = AP_n(x) + BP_{n-1}(x) + CP_{n-2}(x)$, where A, B, C are real numbers with $A \neq 0$ and $S_n(x) = (A_n x + B_n)P_{n-1}(x) + \Gamma_n P_n(x)$, where $A_n \neq 0, B_n$ and Γ_n are real sequences. We find conditions that the polynomials $Q_n(x)$ and $S_n(x)$ are orthogonal, we derive the three-term recurrence relation which is satisfied by each of them and we illustrate the results by some examples. We also give some results concerning the zeros of the polynomials $Q_n(x)$ and $S_n(x)$.

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1. Introduction

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials on the finite or infinite interval $[a, b]$ with respect to a positive weight function. It is well known [5, 10] that:

(i) they satisfy the three-term recurrence relation

$$a_n P_{n+1}(x) + c_n P_{n-1}(x) + b_n P_n(x) = xP_n(x), \quad n \geq 0, \tag{1.1}$$
$$P_{-1}(x) = 0, \quad P_0(x) = 1$$

where a_n, b_n and c_n are real sequences with $a_n > 0$ and $c_n \neq 0$ for $n \geq 0$ and

(ii) their zeros are real, simple and they are located in (a, b).

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Some results concerning linear combinations of orthogonal polynomials are given in [1] and specially results about the zeros of certain linear combinations of Chebyschev polynomials, because under some conditions the eigenfrequencies of one-dimensional systems of masses and springs can be written in terms of the zeros of the above polynomials. Such systems are studied in many fields of engineering, such as mechanical, electrical, physical as models for multistory and highrise buildings, trains, propagation of waves, etc. It is essential to know if their zeros are real and their location.

Also, the linear combination of orthogonal polynomials was connected with quasi-orthogonality (see [4], [3] and references there in), and the location of their zeros was discussed in [1], [2], [3].

In [6] the authors have generated a new family of polynomials relative to a combination of the orthogonal polynomials $P_n(x)$.

In this paper, we first consider a linear combination of three adjacent orthogonal polynomials:

$$Q_n(x) = AP_n(x) + BP_{n-1}(x) + CP_{n-2}(x), \quad n \geq 1, \tag{1.2}$$

where A, B, C are real numbers and $A \neq 0$ and $P_n(x)$ satisfy (1.1) and also we consider a family of polynomials defined by:

$$S_n(x) = (A_n x + B_n)P_{n-1}(x) + \Gamma_n P_n(x), \quad n \geq 0, \tag{1.3}$$

where A_n, B_n, Γ_n and $A_n \neq 0$ are real sequences and $P_n(x)$ satisfy (1.1). In Section 2 we give conditions such that the new polynomial sequences $\{Q_n(x)\}_{n=0}^{\infty}$ and $\{S_n(x)\}_{n=0}^{\infty}$ to be orthogonal and we find the three-term recurrence relation for both of them. The coefficients of the recurrence relations will be expressed via the coefficients of relation (1.1) and A, B, C or A_n, B_n, Γ_n respectively. The results we obtain in this paper generalize some of the results of the paper [6]. In Section 3 we apply the above results using the classical orthogonal polynomials. In Section 4 we give an upper bound of the difference between the k-th zeros of $P_n(x)$ and $Q_n(x)$ or between the k-th zeros of $P_n(x)$ and $S_n(x)$ using inequalities given in [9].

2. The Recurrence Relation for $Q_n(x)$ and $S_n(x)$

We can easily verify that the orthogonal polynomials $P_n(x)$ which satisfy (1.1), satisfy also the equality

$$\begin{pmatrix} P_{n+1}(x) \\ P_n(x) \end{pmatrix} = \begin{pmatrix} \frac{x-b_n}{a_n} & \frac{-c_n}{a_n} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix}. \tag{2.1}$$

By setting $\begin{pmatrix} P_{n+1}(x) \\ P_n(x) \end{pmatrix} = \pi_{n+1}(x)$ and $\begin{pmatrix} \frac{x-b_n}{a_n} & \frac{-c_n}{a_n} \\ 1 & 0 \end{pmatrix} = L_n(x)$, equation (2.1) takes the form:

$$\pi_{n+1}(x) = L_n(x)\pi_n(x). \tag{2.2}$$

Also from the relation (1.2), using (1.1), we obtain

$$Q_{n+1}(x) = [B + \frac{A(x-b_n)}{a_n}]P_n(x) + [C - \frac{Ac_n}{a_n}]P_{n-1}(x) \tag{2.3}$$

and

$$Q_n(x) = [A - \frac{Ca_{n-1}}{c_{n-1}}]P_n(x) + [B + \frac{C(x-b_{n-1})}{c_{n-1}}]P_{n-1}(x). \tag{2.4}$$

By setting

$$\begin{pmatrix} Q_{n+1}(x) \\ Q_n(x) \end{pmatrix} = r_{n+1}(x) \quad \text{and} \quad \begin{pmatrix} \frac{A(x-b_n)}{a_n} + B & C - \frac{Ac_n}{a_n} \\ A - \frac{Ca_{n-1}}{c_{n-1}} & B + \frac{C(x-b_{n-1})}{c_{n-1}} \end{pmatrix} = M_n(x),$$

from (2.3), (2.4) and (2.2) we get

$$r_{n+1}(x) = M_n(x)L_n(x)M_{n-1}^{-1}(x)r_n(x) \tag{2.5}$$

by the assumption that $M_{n-1}^{-1}(x)$ exists, which is satisfied since $A \neq 0$.

We require the polynomial sequence $Q_n(x)$ to be orthogonal in an interval with respect to a positive weight function. A way to succeed this, is to find real sequences $k_n^{(1)}, \lambda_n^{(1)}, \mu_n^{(1)}$ with $\lambda_n^{(1)} > 0$, $\mu_n^{(1)} \neq 0$ for $n \geq 0$ such that the three-term recurrence relation

$$\lambda_n^{(1)} Q_{n+1}(x) + \mu_n^{(1)} Q_{n-1}(x) + k_n^{(1)} Q_n(x) = xQ_n(x), \quad n \geq 0 \tag{2.6}$$

holds. Or setting $\begin{pmatrix} \frac{x-k_n^{(1)}}{\lambda_n^{(1)}} & \frac{-\mu_n^{(1)}}{\lambda_n^{(1)}} \\ 1 & 0 \end{pmatrix} = \Pi_n^{(1)}(x)$, and using (2.6) we obtain:

$$r_{n+1}(x) = \Pi_n^{(1)}(x)r_n(x)\,. \tag{2.7}$$

So, from (2.5) and (2.7) we have to define $k_n^{(1)}, \lambda_n^{(1)}, \mu_n^{(1)}$ such that the equation

$$M_n(x)L_{n-1}(x) = \Pi_n^{(1)}(x)M_{n-1}(x) \tag{2.8}$$

holds.

Similarly, from the relation (1.3), using (1.1), we obtain

$$S_{n+1}(x) = [(A_{n+1}x + B_{n+1}) + \frac{\Gamma_{n+1}(x-b_n)}{a_n}]P_n(x) - \frac{\Gamma_{n+1}c_n}{a_n}P_{n-1}(x). \tag{2.9}$$

By setting

$$\begin{pmatrix} S_{n+1}(x) \\ S_n(x) \end{pmatrix} = t_{n+1}(x),$$

$$\begin{pmatrix} (A_{n+1}x + B_{n+1}) + \frac{\Gamma_{n+1}(x-b_n)}{a_n} & -\Gamma_{n+1}\frac{c_n}{a_n} \\ \Gamma_n & A_nx + B_n \end{pmatrix} = N_n(x),$$

from (1.3), (2.2) and (2.9) we get

$$t_{n+1}(x) = N_n(x)L_n(x)N_{n-1}^{-1}(x)t_n(x) \tag{2.10}$$

by the assumption that $N_{n-1}^{-1}(x)$ exists, which is satisfied since $A_n \neq 0$ for $x \neq 0$.

We also require the polynomial sequence $S_n(x)$ to be orthogonal so we have to evaluate real sequences $k_n^{(2)}, \lambda_n^{(2)}, \mu_n^{(2)}$ with $\lambda_n^{(2)} > 0$, $\mu_n^{(2)} \neq 0$ for $n \geq 0$ such that the three-term recurrence relation

$$\lambda_n^{(2)}S_{n+1}(x) + \mu_n^{(2)}S_{n-1}(x) + k_n^{(2)}S_n(x) = xS_n(x), \quad n \geq 0 \tag{2.11}$$

holds. Or setting again $\begin{pmatrix} \frac{x-k_n^{(2)}}{\lambda_n^{(2)}} & \frac{-\mu_n^{(2)}}{\lambda_n^{(2)}} \\ 1 & 0 \end{pmatrix} = \Pi_n^{(2)}(x)$, and using (2.11) we obtain:

$$t_{n+1}(x) = \Pi_n^{(2)}(x)t_n(x)\,. \tag{2.12}$$

So, from (2.10) and (2.12) we have to define $k_n^{(2)}, \lambda_n^{(2)}, \mu_n^{(2)}$ such that the equation

$$M_n(x)L_{n-1}(x) = \Pi_n^{(2)}(x)M_{n-1}(x) \tag{2.13}$$

holds.

Theorem 2.1. *Let $P_n(x)$ be a sequence of orthogonal polynomials which satisfy (1.1). The linear combination of three adjacent polynomials $Q_n(x) = AP_n(x) + BP_{n-1}(x) + CP_{n-2}(x)$, $A \neq 0$ are orthogonal polynomials if the* *equalities:*

$$AC(a_n - a_{n-2}) + B^2(a_{n-1} - a_n) + AB(b_n - b_{n-1}) + A^2(c_{n-2} - c_n) = 0 \quad (2.14)$$

and

$$AC(b_{n-2} - b_n) + BC(a_n - a_{n-1}) + AB(c_{n-1} - c_{n-2}) = 0 \quad (2.15)$$

hold. Also the three-term recurrence relation which is satisfied by $Q_n(x)$ *is:*

$$a_nQ_{n+1}(x) + c_{n-2}Q_{n-1}(x) + [\frac{B(a_{n-1} - a_n)}{A} + b_n]Q_n(x) = xQ_n(x), \quad (2.16)$$

Proof. From (2.8) it follows:

$$\begin{pmatrix} \frac{A(x-b_n)}{a_n} + B & C - \frac{Ac_n}{a_n} \\ A - \frac{Ca_{n-1}}{c_{n-1}} & B + \frac{C(x-b_{n-1})}{c_{n-1}} \end{pmatrix} \begin{pmatrix} \frac{x-b_{n-1}}{a_{n-1}} & \frac{-c_{n-1}}{a_{n-1}} \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{x-k_n^{(1)}}{\lambda_n^{(1)}} & \frac{-\mu_n^{(1)}}{\lambda_n^{(1)}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{A(x-b_{n-1})}{a_{n-1}} + B & C - \frac{Ac_{n-1}}{a_{n-1}} \\ A - \frac{Ca_{n-2}}{c_{n-2}} & B + \frac{C(x-b_{n-2})}{c_{n-2}} \end{pmatrix}$$

or

$$\begin{pmatrix} \frac{A(x-b_n)(x-b_{n-1})}{a_na_{n-1}} + \frac{B(x-b_{n-1})}{a_{n-1}} + C - \frac{Ac_n}{a_n} & -\frac{A(x-b_n)c_{n-1}}{a_na_{n-1}} - \frac{Bc_{n-1}}{a_{n-1}} \\ \frac{A(x-b_{n-1})}{a_{n-1}} + B - \frac{Ca_{n-1}}{c_{n-1}}\frac{x-b_{n-1}}{a_{n-1}} + \frac{C(x-b_{n-1})}{a_{n-1}} & -\frac{Ac_{n-1}}{a_{n-1}} + \frac{Ca_{n-1}}{c_{n-1}}\frac{c_{n-1}}{a_{n-1}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{A(x-b_{n-1})(x-k_n^{(1)})}{a_{n-1}\lambda_n^{(1)}} + \frac{B(x-k_n^{(1)})}{\lambda_n^{(1)}} + \frac{Ca_{n-2}\mu_n^{(1)}}{\lambda_n^{(1)}c_{n-2}} & \frac{C(x-k_n^{(1)})}{\lambda_n^{(1)}} - \frac{A(x-k_n^{(1)})c_{n-1}}{a_{n-1}\lambda_n^{(1)}} - \frac{B\mu_n^{(1)}}{\lambda_n^{(1)}} - \frac{C(x-b_{n-2})\mu_n^{(1)}}{c_{n-2}\lambda_n^{(1)}} \\ \frac{A(x-b_{n-1})}{a_{n-1}} + B & C - \frac{Ac_{n-1}}{a_{n-1}} \end{pmatrix}. \quad (2.17)$$

Since the elements of the above matrices are polynomials, from (2.17) along with the equality of polynomials, the following equalities must hold:

$$\frac{A}{a_na_{n-1}} = \frac{A}{\lambda_n^{(1)}a_{n-1}}, \quad (2.18)$$

$$-\frac{Ab_{n-1}}{a_na_{n-1}} - \frac{Ab_n}{a_na_{n-1}} + \frac{B}{a_{n-1}} = -\frac{Ak_n^{(1)}}{\lambda_n^{(1)}a_{n-1}} - \frac{Ab_{n-1}}{\lambda_n^{(1)}a_{n-1}} + \frac{B}{\lambda_n^{(1)}}, \quad (2.19)$$

$$\frac{Ab_nb_{n-1}}{a_na_{n-1}}-\frac{Bb_{n-1}}{a_{n-1}}+C-\frac{Ac_n}{a_n}=\frac{Ak_n^{(1)}b_{n-1}}{\lambda_n^{(1)}a_{n-1}}-\frac{Bk_n^{(1)}}{\lambda_n^{(1)}}+\frac{Ca_{n-2}\mu_n^{(1)}}{\lambda_n^{(1)}c_{n-2}}-\frac{Ac_{n-2}}{a_n}, \quad (2.20)$$

$$-\frac{Ac_{n-1}}{a_na_{n-1}}=\frac{C}{\lambda_n^{(1)}}-\frac{Ac_{n-1}}{\lambda_n^{(1)}a_{n-1}}-\frac{C\mu_n^{(1)}}{\lambda_n^{(1)}c_{n-2}}, \quad (2.21)$$

$$\frac{Ab_nc_{n-1}}{a_na_{n-1}}-\frac{Bc_{n-1}}{a_{n-1}}=-\frac{Ck_n^{(1)}}{\lambda_n^{(1)}}+\frac{Ak_n^{(1)}c_{n-1}}{\lambda_n^{(1)}a_{n-1}}-\frac{B\mu_n^{(1)}}{\lambda_n^{(1)}}+\frac{C\mu_n^{(1)}b_{n-2}}{\lambda_n^{(1)}c_{n-2}}. \quad (2.22)$$

From (2.18) we obtain

$$\lambda_n^{(1)}=a_n\,. \quad (2.23)$$

From (2.21) and (2.19) using (2.23) we obtain respectively

$$\mu_n^{(1)}=c_{n-2} \quad (2.24)$$

and

$$k_n^{(1)}=\frac{B(a_{n-1}-a_n)}{A}+b_n. \quad (2.25)$$

Also, from (2.20) and (2.22) using (2.23), (2.24) and (2.25) we obtain respectively the equalities (2.14) and (2.15). The three-term recurrence relation (2.16) becomes immediately from (2.6) taking in account (2.23), (2.24) and (2.25).

Corollary 2.1. *If $C=0$, the polynomials*

$$R_n(x)=AP_n(x)+BP_{n-1}(x), \quad A\neq 0\,, \quad (2.26)$$

form an orthogonal sequence of polynomials, if the equality

$$AB(b_{n-1}-b_n)+B^2(a_n-a_{n-1})+A^2(c_n-c_{n-1})=0 \quad (2.27)$$

holds and the three-term recurrence relation becomes:

$$a_nR_{n+1}(x)+c_{n-1}R_{n-1}(x)+[\frac{B(a_{n-1}-a_n)}{A}+b_n]R_n(x)=xR_n(x). \quad (2.28)$$

Proof. It follows easily from the equalities (2.18)-(2.22), by setting $C=0$.

Theorem 2.2. *Let $P_n(x)$ be a sequence of orthogonal polynomials which satisfy (1.1). The polynomials $S_n(x)=(A_nx+B_n)P_{n-1}(x)+\Gamma_nP_n(x)$, $A_n\neq 0$ are orthogonal if the equalities:*

$$[\frac{M_{n+1}}{\Lambda_{n+1}}-\frac{M_n}{\Lambda_n}][A_nb_{n-1}+B_n]=\frac{\Gamma_{n+1}\Lambda_n}{\Lambda_{n+1}}c_n-\frac{\Gamma_{n-1}A_n}{A_{n-1}}c_{n-1}\,, \quad (2.29)$$

$$\frac{M_{n+1}}{\Lambda_{n+1}}\frac{M_n}{\Lambda_n}=\frac{B_n}{A_n}-\frac{B_{n-1}}{A_{n-1}} \quad (2.30)$$

hold, where

$$\Lambda_n=A_na_{n-1}+\Gamma_n \quad (2.31)$$

and

$$M_n = \Gamma_n b_{n-1} - B_n a_{n-1}. \tag{2.32}$$

Also the three-term recurrence relation which is satisfied by $S_n(x)$ is:

$$\lambda_n^{(2)} S_{n+1}(x) + \mu_n^{(2)} S_{n-1}(x) + k_n^{(2)} S_n(x) = x S_n(x), \tag{2.33}$$

where the coefficients are given by

$$\lambda_n^{(2)} = \frac{\Lambda_n a_n}{\Lambda_{n+1}}, \tag{2.34}$$

$$\mu_n^{(2)} = \frac{A_n}{A_{n-1}} c_{n-1} \tag{2.35}$$

and

$$k_n^{(2)} = \frac{M_{n+1}}{\Lambda_{n+1}} - \frac{M_n}{\Lambda_n} + b_{n-1}. \tag{2.36}$$

Proof. From (2.13) using exactly the same procedure as in the proof of Theorem 2.1, we obtain that the following equalities must hold:

$$\frac{A_{n+1}}{a_{n-1}} + \frac{\Gamma_{n+1}}{a_n a_{n-1}} = \frac{A_n}{\lambda_n^{(2)}} + \frac{\Gamma_n}{a_{n-1}\lambda_n^{(2)}} \tag{2.37}$$

$$-\frac{A_{n+1} b_{n-1}}{a_{n-1}} + \frac{B_{n+1}}{a_{n-1}} - \frac{\Gamma_{n+1} b_n}{a_n a_{n-1}} - \frac{\Gamma_{n+1} b_{n-1}}{a_n a_{n-1}} = -\frac{A_n k_n^{(2)}}{\lambda_n^{(2)}} + \frac{B_n}{\lambda_n^{(2)}} - \frac{\Gamma_n b_{n-1}}{a_{n-1}\lambda_n^{(2)}} - \frac{\Gamma_n k_n^{(2)}}{a_{n-1}\lambda_n^{(2)}} \tag{2.38}$$

$$-\frac{B_{n+1} b_{n-1}}{a_{n-1}} + \frac{\Gamma_{n+1} b_n b_{n-1}}{a_n a_{n-1}} - \frac{\Gamma_{n+1} c_n}{a_n} = -\frac{B_n k_n^{(2)}}{\lambda_n^{(2)}} + \frac{\Gamma_n k_n^{(2)} b_{n-1}}{a_{n-1}\lambda_n^{(2)}} - \frac{\Gamma_{n-1}\mu_n^{(2)}}{\lambda_n^{(2)}} \tag{2.39}$$

$$-\frac{A_{n+1} c_{n-1}}{a_{n-1}} - \frac{\Gamma_{n+1} c_{n-1}}{a_n a_{n-1}} = -\frac{\Gamma_n c_{n-1}}{a_{n-1}\lambda_n^{(2)}} - \frac{A_{n-1}\mu_n^{(2)}}{\lambda_n^{(2)}} \tag{2.40}$$

$$-\frac{B_{n+1} c_{n-1}}{a_{n-1}} + \frac{\Gamma_{n+1} b_n c_{n-1}}{a_n a_{n-1}} = \frac{\Gamma_n k_n^{(2)} c_{n-1}}{a_{n-1}\lambda_n^{(2)}} - \frac{B_{n-1}\mu_n^{(2)}}{\lambda_n^{(2)}}. \tag{2.41}$$

So, from (2.37), using (2.31) we obtain (2.34), from (2.40) using (2.31) and (2.34) we obtain (2.35), from (2.38) using (2.31), (2.32) and (2.34) we obtain (2.36) and from (2.39) and (2.40) using (2.34), (2.35) and (2.36) we obtain (2.29) and (2.30) respectively.

Remark 2.1. For the special case $B_n = 0$ and $\Gamma_n = 1 - A_n$, Theorem 2.2 is the same as the one which has been proved in [6].

Corollary 2.2. *Using Theorem 2.2 for $\Gamma_n = 0$ we obtain that the polynomial sequence $S_n(x) = (A_n x + B_n)P_{n-1}(x)$ is orthogonal if:*

(i)$B_n = 0$, or

(ii) $\frac{B_n}{A_n} = -b_{n-1}$ and the relation $b_n b_{n-1} = b_{n-2} - b_{n-1}$ holds.

In the first case using (2.34) we obtain that the polynomials $S_n(x) = A_n x P_{n-1}(x)$ satisfy the recurrence relation:

$$\frac{A_n a_{n-1}}{A_{n+1}} S_{n+1}(x) + \frac{A_n c_{n-1}}{A_{n-1}} S_{n-1}(x) + b_{n-1} S_n(x) = x S_n(x) \tag{2.42}$$

and in the second case using (2.34) we obtain that the polynomials $S_n(x) = A_n(x - b_{n-1})P_{n-1}(x)$ satisfy the recurrence relation:

$$\frac{A_n a_{n-1}}{A_{n+1}} S_{n+1}(x) + \frac{A_n c_{n-1}}{A_{n-1}} S_{n-1}(x) + b_n S_n(x) = x S_n(x). \tag{2.43}$$

Corollary 2.3. *For $A_n = A \neq 0$, $B_n = B$ and $\Gamma_n = \Gamma$ real numbers, using Theorem 2.2, we obtain that the polynomials $S_n(x) = (Ax+B)P_{n-1}(x) + \Gamma P_n(x)$ are orthogonal if:*

(i) $\Gamma = 0$ and $B = 0$, or

(ii) $\frac{Aa_{n-1}+\Gamma}{Aa_n+\Gamma} = \frac{c_{n-1}}{c_n}$ holds.

In the first case using (2.34) we obtain that the polynomials $S_n(x) = AxP_{n-1}(x)$ satisfy the recurrence relation:

$$a_{n-1} S_{n+1}(x) + c_{n-1} S_{n-1}(x) + b_{n-1} S_n(x) = x S_n(x) \tag{2.44}$$

which becomes also from (2.42) for $A_n = A$ and in the second case using (2.34) we obtain that the polynomials $S_n(x) = (Ax + B)P_{n-1}(x) + \Gamma P_n(x)$ satisfy the recurrence relation:

$$\frac{a_n c_{n-1}}{c_n} S_{n+1}(x) + c_{n-1} S_{n-1}(x) + b_{n-1} S_n(x) = x S_n(x). \tag{2.45}$$

3. Examples

1) The linear combination of three or two adjacent Chebyschev polynomials of first or second kind, is also a member of Chebyshev polynomials.

It is obvious, because the Chebyschev polynomials satisfy the relation (1.1) for $a_n = c_n = \frac{1}{2}$ and $b_n = 0$ [5]. For every real number A, B, C, the equations (2.14) and (2.15) or (2.27) are satisfied and (2.16) becomes exactly the same as

(1.1) for Chebyshev polynomials.

2) The Legendre polynomials $P_n(x)$ satisfy (1.1) with $a_n = \frac{n+1}{2n+1}$, $b_n = 0$ and $c_n = \frac{n}{2n+1}$ [5]. The equalities (2.14) and (2.15) become:

$$(2B^2 - 4AC - 4)n + (2AC - 3B^2 + 2A^2) = 0$$

and

$$B[(2A - 2C)n + A + 3C] = 0\,,$$

respectively, from which we obtain $B = 0$ and $C = -A$. So the polynomials $Q_n(x) = A[P_n(x) - P_{n-2}(x)]$ are orthogonal and satisfy the following relation:

$$\frac{n+1}{2n+1}Q_{n+1}(x) + \frac{n-2}{2n-3}Q_{n-1}(x) = xQ_n(x). \tag{3.1}$$

Also (2.27) gives $\frac{B}{A} = \pm 1$ so the polynomials $R_n(x) = A[P_n(x) \pm P_{n-1}(x)]$ are orthogonal and satisfy the relation:

$$\frac{n+1}{2n+1}R_n(x) + \frac{n-1}{2n-1}R_{n-1}(x) \pm \frac{1}{(2n+1)(2n-1)}R_n(x) = xR_n(x). \tag{3.2}$$

3) The Laguerre polynomials $L_n^\alpha(x)$ satisfy (1.1) with $a_n = n+1$, $b_n = -(2n+\alpha+1)$ and $c_n = n+\alpha$ [5]. The equalities (2.14) and (2.15) become:

$$2AC - B^2 - 2AB - 2A^2 = 0$$

and

$$4AC + BC + AB = 0$$

respectively, from which we obtain $B = -2A$ and $C = A$. So the polynomials $Q_n(x) = A[L_n^\alpha(x) - 2L_{n-1}^\alpha(x) + L_{n-2}^\alpha(x)]$ are orthogonal and satisfy the recurrence relation:

$$(n+1)Q_{n+1}(x) + (n+\alpha-2)Q_{n-1}(x) - (2n+\alpha-1)Q_n(x) = xQ_n(x) \tag{3.3}$$

which is the recurrence relation for the orthogonal Laguerre polynomials $L_n^{\alpha-2}(x)$.

Also, from (2.27) we obtain $\frac{B}{A} = -1$, so the polynomials $R_n(x) = A[L_n^\alpha(x) - L_{n-1}^\alpha(x)]$ are orthogonal and (2.28) becomes:

$$(n+1)R_{n+1}(x) + (n+\alpha-1)R_{n-1}(x) - (2n+\alpha)R_n(x) = xR_n(x), \tag{3.4}$$

which is the three-term recurrence relation for the orthogonal Laguerre polynomials $L_n^{\alpha-1}(x)$. This is a known result, see for example [10].

4) The Ultraspherical polynomials $C_n^\lambda(x)$ satisfy (1.1) with $a_n = \frac{n+1}{2(\lambda+n)}$, $b_n = 0$ and $c_n = \frac{2\lambda+n-1}{2(\lambda+n)}$ [5]. The equalities (2.14) and (2.15) become:

$$[2A(C+1) + B^2]n + 2A(C+A)(\lambda-1)^2 + B^2(\lambda-2) = 0$$

and

$$B[[C(\lambda-1)+A(\lambda+1)]n+\lambda(\lambda+1)A+(\lambda-1)(\lambda-2)C]=0$$

from which we obtain $B=0$ and $C=-A$, so the polynomials $Q_n(x)=A[C_n^\lambda(x)-C_{n-2}^\lambda(x)]$ are orthogonal and (2.16) becomes

$$\frac{n+1}{2(\lambda+n)}Q_{n+1}(x)+\frac{2\lambda+n-3}{2(\lambda+n-2)}Q_{n-1}(x)=xQ_n(x). \tag{3.5}$$

Also from (2.27), we get $\frac{B}{A}=\pm1$ and $\lambda>\frac{1}{2}$, or $\lambda=1$ and A,B is an arbitrary real number, so in the first case, from (2.28) we obtain the relation:

$$\frac{n+1}{2(\lambda+n)}R_n(x)+\frac{2\lambda+n-2}{2(\lambda+n-1)}R_{n-1}(x)\pm(1-\lambda)R_n(x)=xR_n(x), \tag{3.6}$$

for the orthogonal polynomials $R_n(x)=A[C_n^\lambda(x)\pm C_{n-1}^\lambda(x)]$, $\lambda>1/2$ and in the second case the polynomials $R_n(x)=AC_n^1(x)+BC_{n-1}^1(x)$ are the Chebyshev polynomials and were examined in the first example.

5) Corollary 2.2 holds for all the classical orthogonal polynomials $P_n(x)$ in the first case where $B_n=0$, or for the classical polynomials whose the coefficients $b_n=0$ in the recurrence (1.1).

6) The second case of Corollary 2.3 holds for Chebyshev polynomials $T_n(x)$ and the recurrence which the polynomials $S_n(x)=(Ax+B)T_{n-1}(x)+\Gamma T_n(x)$ satisfy is the same recurrence as the polynomials $T_n(x)$. It also holds for Laguerre polynomials $L_n^a(x)$ when $\Gamma=A(a-1)$. It follows from the relation $\frac{Aa_{n-1}+\Gamma}{Aa_n+\Gamma}=\frac{c_{n-1}}{c_n}$ for $a_n=n+1$ and $c_n=n+a$. So using (2.45) the polynomials $S_n(x)=(Ax+B)L_{n-1}^a(x)+A(a-1)L_n^a(x)$ satisfy the recurrence:

$$\frac{(n+1)(n+a-1)}{n+a}S_{n+1}(x)+(n+a-1)S_{n-1}(x)-(2n+a-1)S_n(x)=xR_n(x). \tag{3.7}$$

4. Some Results about the Zeros of $Q_n(x)$ and $S_n(x)$

The zeros of the orthogonal polynomials $P_n(x)$ which satisfy (1.1) are the same as the zeros of the corresponding orthonormal polynomials $\tilde{P}_n(x)$ which satisfy [7] the following recurrence relation:

$$\sqrt{a_nc_{n+1}}\tilde{P}_{n+1}(x)+\sqrt{a_{n-1}c_n}\tilde{P}_{n-1}(x)+b_n\tilde{P}_n(x)=x\tilde{P}_n(x) \tag{4.1}$$

and they are also the eigenvalues of a tridiagonal operator T, defined by

$$Te_n=\sqrt{a_nc_{n+1}}e_{n+1}+\sqrt{a_{n-1}c_n}e_{n-1}+b_ne_n, \quad n\geq1, \tag{4.2}$$

where e_n is an orthonormal basis in a Hilbert space H.

In [7, 8] have been found assumptions such that the specrtum of T to be the entire interval $[-\sqrt{ac}+b, \sqrt{ac}+b]$, where $a=\lim_{n\to\infty} a_n$, $c=\lim_{n\to\infty} c_n$ and $b=\lim_{n\to\infty} b_n$.

The orthonormal polynomials $\tilde{Q}_n(x)$ and $\tilde{S}_n(x)$ satisfy the recurrence relations

$$\sqrt{a_n c_{n-1}}\tilde{Q}_{n+1}(x)+\sqrt{a_{n-1}c_{n-2}}\tilde{Q}_{n-1}(x)+[\frac{B(a_{n-1}-a_n)}{A}+b_n]\tilde{Q}_n(x) = x\tilde{Q}_n(x) \tag{4.3}$$

and

$$\sqrt{\lambda_n^{(2)}\mu_{n+1}^{(2)}}\tilde{S}_{n+1}(x)+\sqrt{\lambda_{n-1}^{(2)}\mu_n^{(2)}}\tilde{S}_{n-1}(x)+k_n^{(2)}\tilde{S}_n(x)=x\tilde{S}_n(x) \tag{4.4}$$

with $\lambda_n^{(2)}$, $\mu_n^{(2)}$ and $k_n^{(2)}$ are given by (2.34), (2.35) and (2.36) and the corresponding tridiagonal operators are

$$T_1 e_n=\sqrt{a_n c_{n-1}}e_{n+1}+\sqrt{a_{n-1}c_{n-2}}e_{n-1}+[\frac{B(a_{n-1}-a_n)}{A}+b_n]e_n \tag{4.5}$$

and

$$T_2 e_n=\sqrt{\lambda_n^{(2)}\mu_{n+1}^{(2)}}e_{n+1}+\sqrt{\lambda_{n-1}^{(2)}\mu_n^{(2)}}e_{n-1}+k_n^{(2)}e_n. \tag{4.6}$$

It is obvious that $\lim_{n\to\infty}\sqrt{a_n c_{n-1}}=\lim_{n\to\infty}\sqrt{a_n c_{n+1}}=\sqrt{ac}$, and

$$\lim_{n\to\infty}[\frac{B(a_{n-1}-a_n)}{A}+b_n]=\lim_{n\to\infty} b_n=b.$$

Also, if $\lim_{n\to\infty}\frac{\Gamma_n}{A_n}$ and $\lim_{n\to\infty}\frac{B_n}{A_n}$ exist and are finite, then using the relations (2.31), (2.32), (2.34), (2.35) and (2.36) we obtain that $\lim_{n\to\infty}\sqrt{\lambda_n^{(2)}\mu_{n+1}^{(2)}}= \lim_{n\to\infty}\sqrt{a_n c_n}=\lim_{n\to\infty}\sqrt{a_n c_{n-1}}=\sqrt{ac}$ and $\lim_{n\to\infty} k_n^{(2)}=\lim_{n\to\infty} b_n=b$, so it follows:

Proposition 4.1. *The operators T, T_1 and T_2 defined by (4.2), (4.5) and (4.6) respectively, have the same spectrum, so the support of the measure of orthogonality of the polynomials $P_n(x)$, $Q_n(x)$ and $S_n(x)$ is the same.*

Let p_{nk} and q_{nk}, $k=1,....,n$ be the positive zeros of the orthogonal polynomials $P_n(x)$ and $Q_n(x)$ correspondingly. In [9] have been found bounds for the zeros of the orthogonal polynomials. Using the analogous notation, we obtain:

$$0\le p_{nk}\le 2\max_k\sqrt{a_k c_{k+1}}+\max_k|b_k|, \quad k=1,...,n\,, \tag{4.7}$$

and

$$0\le q_{nk}\le 2\max_k\sqrt{a_k c_{k-1}}+\max_k|\frac{B(a_{k-1}-a_k)}{A}+b_k|, \quad k=1,...,n. \tag{4.8}$$

Combining (4.7) and (4.8) we have:

$$|q_{nk} - p_{nk}| \leq 2 \max_k \sqrt{a_k c_{k-1}} + \max_k |\frac{B(a_{k-1} - a_k)}{A}| + \max_k |b_k|, \qquad (4.9)$$

$k = 1, ..., n$, if the sequence c_k decreases and

$$|q_{nk} - p_{nk}| \leq 2 \max_k \sqrt{a_k c_{k+1}} + \max_k |\frac{B(a_{k-1} - a_k)}{A}| + \max_k |b_k|, \qquad (4.10)$$

$k = 1, ..., n$, if the sequence c_k increases.

Finally from (4.9) and (4.10) we obtain the following proposition:

Propotision 4.2. *An upper bound of the difference between the k-th zero of the orthogonal polynomial $P_n(x)$ and $Q_n(x)$ is:*

$$|q_{nk} - p_{nk}| \leq 2\sqrt{c_1} \max_k \sqrt{a_k} + \frac{2|B|}{|A|} \max_k |a_k| + \max_k |b_k|, \quad k = 1, ..., n, \quad (4.11)$$

or

$$|q_{nk} - p_{nk}| \leq 2\sqrt{c_1 a_1} + \frac{2|B|}{|A|} a_1 + \max_k |b_k|, \quad k = 1, ..., n, \qquad (4.12)$$

if c_k decreases and a_k increases or a_k decreases and

$$|q_{nk} - p_{nk}| \leq 2 \max_k \sqrt{a_k c_{k+1}} + \frac{2|B|}{|A|} \max_k |a_k| + \max_k |b_k|, \quad k = 1, ..., n, \quad (4.13)$$

or

$$|q_{nk} - p_{nk}| \leq 2\sqrt{a_1} \max_k \sqrt{c_{k+1}} + \frac{2|B|}{|A|} a_1 + \max_k |b_k|, \quad k = 1, ..., n, \qquad (4.14)$$

if c_k increases and a_k increases or a_k decreases.

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