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ON SOLVABILITY OF NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we consider neutral stochastic functional differential equations with unbounded delay in a Hilbert space. Existence and uniqueness of solutions of such equations are established using local Lipschitz conditions on the nonlinear terms exploiting the contraction mapping principle. An example is given to illustrate the theory.

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1. Introduction

Consider the neutral stochastic functional differential equation with an unbounded delay in a Hilbert space of the form:

$$d[x(t)+f(t,x_t)] = a(t,x_t)dt + b(t,x_t)dw(t), \quad 0 \le t \le T; \tag{1}$$

$$x(t) = \varphi(t), \quad t \le 0; \tag{2}$$

where $x_t(s) = x(s)$ for $s \in (-\infty, t]$ and the equation will be made precise in Section 2. This class of equations was originally introduced by J.K. Hale in the deterministic case (that is, $b \equiv 0$) in infinite dimensions with a finite delay, see Hale and Verduyn Lunel [3]. Motivated by the theory of aeroelasticity,

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Kolmanovskii and Nosov [5] introduced equation (1) in finite dimensions with a finite delay unlike in (2) by using a global Lipschitz condition on $a(t,u)$ and $b(t,u)$ while assuming $f(t,u)$ to be a contraction mapping (Lipschitz constant $q<1$). Rodkina [6] studied several existence and uniqueness results concerning the problem (1)–(2) in finite dimensions by exploiting a linear growth and Hölder type conditions on $a(t,u)$ and $b(t,u)$ while assuming $f(t,u)$ to be a contraction. The results are however claimed to be valid in a Hilbert space in [6]. Quite recently, the author [1] generalized the principal result from [6] in an infinite dimensional setting by using a nonlinear growth condition on $a(t,u)$ and $b(t,u)$ while considering all the other conditions to be the same as in [6]. The main result obtained in [1] was applied to a problem on lossless transmission lines.

In this note, our objective is to relax the condition on $f(t,u)$. To be precise, we shall study the existence and uniqueness of a solution of the problem (1)–(2) by using only a local Lipschitz condition on $a(t,u)$, $b(t,u)$ and $f(t,u)$ instead of the global conditions imposed in all the aforementioned works. The result obtained appears to be new even in the finite delay case.

We shall discuss now, though, briefly the methods employed in the literature in studying the problem (1)–(2). In [5], the authors considered the existence problem by using the Banach's contraction mapping principle. Rodkina [6] studied the existence problem by using the method of successive approximations. The nonlinear growth condition assumed in [1] compelled the use of a comparison principle together with the successive approximation procedure. These methods, though, general do not seem to permit the relaxation of the contraction condition $q<1$. We give an example in Section 4 below where this condition does not hold. This motivates the relaxation of this condition by a local condition.

In the next section, we give the preliminaries. Section 3 addresses the main result on the existence and uniqueness of a global solution. An example is included in Section 4.

2. Preliminaries

Let X and Y be real separable Hilbert spaces and $L(Y,X)$ be the space of bounded linear operators mapping Y into X. For convenience, we shall use $|\cdot|$ to denote norms in X,Y and $L(Y,X)$. Let $w(t)\in Y$ be the standard Wiener process defined on the probability space (Ω,U,P); the σ–algebra $B_{t_1,t_2}(dw)$ is the minimal subalgebra of the algebra U with respect to which the random variables $w(s)-w(t)$ are measurable for all $t_1 \le t < s \le t_2$. Let the σ–algebra $B_{-\infty,0}(\varphi)$ (that is, the minimal subalgebra of the algebra U with respect to which a process φ is measurable) not be dependent on $B_{0,\varphi}(dw)$.

Definition 1. The X–valued random process $x(t)$ is called the solution of the problem (1)–(2) in $(-\infty, T)$, $T>0$, if the σ–algebra $B_{t,\infty}(dw)$ for each $t>0$ does not depend on $B_{-\infty,t}(x)B_{0,t}(dw)$ and with probability 1 the equality

$$x(t) = f(t,x_t) + \varphi(0) - f(0,\varphi_0) + \int_0^t a(s,x_s)ds + \int_0^t b(s,x_s)dw(s), \quad 0 \le t \le T,$$

holds; here $x(t)=\varphi(t)$ for $t \ge 0$. The second integral is understood in the sense of Itô.

Let C_T be the space of continuous functions $x : (-\infty, T] \to X$ with norm $||x||_T = \sup\limits_{s \le T} |x(s)|$. The initial process φ has almost surely continuous paths and $E||\varphi(\cdot,w)||_0^p < \infty$, where $p \ge 2$. We now make equation (1) precise. Let the functions $a(t,u)$, $f(t,u)$ and $b(t,u)$ be defined as follows: $a : [0,T]\times C_t \to X$, $f : [0,T]\times C_t \to X$ and $b : [0,T]\times C_t \to L(Y,X)$ are Borel measurable; and for each (t,u) are measurable with respect to the σ–algebra $B_{0,t}(dw)$.

Let the following assumptions hold a.s.:

(A1) For $p \ge 2$, the continuous functions $f(t,u)$, $a(t,u)$ and $b(t,u)$ are such that for every $T>0$, there exist positive constants $C_i = C_i(T)$, $i=1,2,3$ such that

$$|a(t,x_t)-a(t,y_t)|^p \le C_1||x-y||_t^p; \quad |b(t,x_t)-b(t,y_t)|^p \le C_2||x-y||_t^p;$$
$$|f(t,x_t)-f(t,y_t)|^p \le C_3||x-y||_t^p.$$

(A2) For $p \ge 2$, $f(t,u)$ is continuous in the p-th–mean sense:

$$E|f(t+h,x_{t+h})-f(t,x_t)|^p \to 0 \quad \text{as} \quad h \to 0.$$

Under Assumption (A1), it can be shown that there exist positive constants $C_i = C_i(T)$, $i=4,5$ such that

$$|a(t,x_t)|^p + |b(t,x_t)|^p \le C_4(1+||x||_t^p); \quad |f(t,x_t)|^p \le C_5(1+||x||_t^p).$$

3. Existence and Uniqueness of a Solution

Theorem 1. *Suppose that Assumptions (A1)–(A2) are satisfied. Then, there exists a time* $0 < T_m = T_{\max} \leq \infty$ *such that the problem (1), (2) has a unique solution. Further, if* $T_m < \infty$*, then* $\lim_{t \uparrow T_m} E|x(t)|^p = \infty$*,* $p \geq 2$*.*

Assume $T > 0$ is a fixed time. Let Γ_T be a closed subspace of $C = C((-\infty, T], L^p(\Omega, X))$, $p \geq 2$ consisting of measurable and $B_{0,t}(dw)$–adapted processes $x = \{x(t), t \in (-\infty, T]\}$ such that $x(t) = \varphi(t), t \leq 0$. We introduce a norm in Γ_T as

$$||x||_{\Gamma_T} := \sup_{0 \leq t \leq T} e^{-bt}(E||x||_t^p)^{1/p}, \quad b > 0;$$

which is clearly equivalent to the norm of C, see [4].

Define a map G on Γ_T: $(0 < t < T)$

$$(Gx)(t) = \varphi(0) - f(0, \varphi_0) + f(t, x_t) + \int_0^t a(s, x_s)ds + \int_0^t b(s, x_s)dw(s),$$

$$(Gx)(t) = \varphi(t), \quad t \leq 0.$$

To prove this theorem, we need some lemmas.

Lemma 1. *For arbitrary* $x \in \Gamma_T$*,* $(Gx)(t)$ *is continuous on* $[0, T]$ *in the* L^p*–sense.*

Proof. Let $0 \leq t \leq T$, $h > 0$ and $t + h \in [0, T]$. Consider

$$(Gx)(t+h) - (Gx)(t) = f(t+h, x_{t+h}) - f(t, x_t)$$
$$+ \int_0^{t+h} a(s, x_s)ds - \int_0^t a(s, x_s)ds + \int_0^{t+h} b(s, x_s)dw(s) - \int_0^t b(s, x_s)dw(s).$$

By Assumption (A2),

$$E|f(t+h, x_{t+h}) - f(t, x_t)|^p \to 0 \quad \text{as} \quad h \to 0.$$

Next, by Assumption (A1), we have

$$E\left|\int_0^{t+h} a(s, x_s)ds - \int_0^t a(s, x_s)ds\right|^p \leq h^{p-1}E\int_t^{t+h} |a(s, x_s)|^p ds$$
$$\leq h^{p-1}C_4 \int_t^{t+h} (E||x||_s^p + 1)ds \leq C_4 h^p \left(\sup_{0 \leq t \leq T} E||x||_t^p + 1\right),$$

and by Proposition 1.9 [4],

$$E\left|\int_0^{t+h} b(s, x_s)dw(s) - \int_0^t b(s, x_s)dw(s)\right|^p = E\left|\int_t^{t+h} b(s, x_s)dw(s)\right|^p$$

$$\leq \left[\frac{p(p-1)}{2}\right]^{p/2} h^{p/2-1} \int_t^{t+h} E|b(s,x_s)|^p ds$$
$$\leq \left[\frac{p(p-1)}{2}\right]^{p/2} h^{p/2} C_4 \left(\sup_{0\leq t\leq T} E||x||_t^p + 1\right).$$

Letting $h \to 0$,

$$E|(Gx)(t+h) - (Gx)(t)|^p \to 0$$

from all the foregoing estimates. One can similarly show that

$$E|(Gx)(t-h) - (Gx)(t)|^p \to 0$$

as $h \to 0$, proving the result.

Lemma 2. *G maps* Γ_T *into itself, that is,* $G(\Gamma_T) \subset \Gamma_T$.

Proof. Let $x \in \Gamma_T$. Then

$$e^{-bt} E|f(0,\varphi_0)|^p \;\leq\; C_5 e^{-bt}\left[E||\varphi||_0^p \,+\, 1\right] \;\leq\; C_5\left[\sup_{0\leq t\leq T} e^{-bt} E||\varphi||_0^p \,+\, 1\right].$$

Similarly, we have

$$e^{-bt} E|f(t,x_t)|^p \leq C_5\left[\sup_{0\leq t\leq T} e^{-bt} E||x||_t^p + 1\right].$$

Next,

$$e^{-bt} E\left|\int_0^t a(s,x_s)ds\right|^p \leq t^{p-1} e^{-bt} C_4 \int_0^t [E||x||_s^p + 1]ds$$
$$\leq T^{2p-1} C_4 \left[\sup_{0\leq t\leq T} e^{-bt} E||x||_t^p + 1\right].$$

Lastly,

$$e^{-bt} E\left|\int_0^t b(s,x_s)dw(s)\right|^p \leq \left[\frac{p(p-1)}{2}\right]^{p/2} t^{p/2-1} e^{-bt} \int_0^t E|b(s,x_s)|^2 ds$$
$$\leq \left[\frac{p(p-1)}{2}\right]^{p/2} T^{p/2} C_4 \left[\sup_{0\leq t\leq T} e^{-bt} E||x||_t^p + 1\right].$$

It follows from all the above estimates that

$$\sup_{0\leq t\leq T} e^{-bt}\left(E|(Gx)(t)|^p\right)^{1/p} < \infty.$$

Proof of Theorem 1. Let $x, y \in \Gamma_T$. Consider

$$(Gx)(t) - (Gy)(t) = f(t,x_t) - f(t,y_t) + \int_0^t [a(s,x_s) - a(s,y_s)]ds$$

$$+\int_0^t [b(s,x_s)-b(s,y_s)]dw(s) = \sum_{i=1}^{3} I_i, \quad \text{say.}$$

Then,

$$e^{-bt}\Big[E|I_1|^p\Big]^{1/p} \le e^{-bt}C_3[E||x-y||_t^p]^{1/p} \le C_3||x-y||_{\Gamma_T}.$$

Next,

$$\begin{aligned}
e^{-bt}[E|I_2|^p]^{1/p} &\le C_1 e^{-bt}\Big\{\Big[\int_0^t E||x-y||_s ds\Big]^p\Big\}^{1/p} \\
&\le C_1 t^{(p-1)/p}\Big[\int_0^t e^{-pbt}E||x-y||_s^p ds\Big]^{1/p} \\
&\le C_1 T^{(p-1)/p}\Big[\int_0^t e^{-pb(t-s)}ds\Big]^{1/p}\Big[\sup_{0\le t\le T} e^{-pbt}E||x-y||_t^p\Big]^{1/p} \\
&\le C_1\frac{T^{(p-1)/p}}{(pb)^{1/p}}\sup_{0\le t\le T} e^{-bt}\Big[E||x-y||_t^p\Big]^{1/p}.
\end{aligned}$$

Similarly, for the stochastic integral, we get

$$\begin{aligned}
e^{-bt}[E|I_3|^p]^{1/p} &\le C_2\Big[\frac{p(p-1)}{2}\Big]^{1/2} T^{1/2-1/p}e^{-bt}\Big[\int_0^t E||x-y||_s^p ds\Big]^{1/p} \\
&\le \frac{C_2}{(pb)^{1/p}}\Big[\frac{p(p-1)}{2}\Big]^{1/2} T^{1/2-1/p}\sup_{0\le t\le T} e^{-bt}\Big[E||x-y||_t^p\Big]^{1/p}.
\end{aligned}$$

Combining all the above estimates, we obtain

$$||G_x - G_y||_{\Gamma_T} = \sup_{0\le t\le T} e^{-bt}[E|(Gx)(t) - (Gy)(t)|^p]^{1/p} \le L||x-y||_{\Gamma_T},$$

where

$$L = C_3 + C_1\frac{T^{(p-1)/p}}{(pb)^{1/p}} + \frac{C_2}{(pb)^{1/p}}\Big[\frac{p(p-1)}{2}\Big]^{1/2}T^{1/2-1/p}.$$

Hence, for $b>0$ sufficiently large and $T>0$ sufficiently small, G is a contraction on Γ_T. Therefore, G has a unique fixed point $x \in \Gamma_T$ and this fixed point is the unique solution of problem (1), (2) on $[0,T]$. Next, we continue the solution for $t \ge T = T_1$, say, see [2] and the references therein. For $t \in [T_1, T_2]$, we say that a function $\hat{x}(t)$ is a continuation of $x(t)$ to the interval $[T_1, T_2]$ if:

a) $\hat{x} \in C((-\infty, T_2], L^p(\Omega, X))$, and

b) $\hat{x}(t) = \varphi(T_1) - f(0,\varphi_{T_1}) + f(t,x_t) + \int_{T_1}^t a(s,x_s)ds + \int_{T_1}^t b(s,x_s)dw(s)$,

a.s.

The continuation of solution applied to $\hat{x}(t)$ is justified by the observation that if we define a new funtion $v(t)$ on $[0, T_2]$ by setting

$$v(t) = \begin{cases} x(t) & \text{if} \quad 0 \leq t \leq T_1, \\ \hat{x}(t) & \text{if} \quad T_1 \leq t \leq T_2, \end{cases}$$

and $v(t) = \varphi(t), t \leq 0$, then $v(t)$ is a solution of problem (1), (2) on $[0, T_2]$. The existence and uniqueness of the continuation $\hat{x}(t)$ is demonstrated exactly as above with only some minor changes. Repeating this procedure, one continues the solution till the time $T_m = T_{\max}$, where $[0, T_m]$ is the maximum interval of the existence and uniqueness of a solution. For T_m finite, $\lim E|x(t)|^p = \infty$ as $t \uparrow T_m$. If not, there exists a sequence $\{\tau_n\}$ converging to T_m and a finite positive number δ such that $E|x(\tau_n)|^p \leq \delta$ for all n. Taking n sufficiently large so that τ_n is infinitesimally close to T_m, one can use the previous arguments to extend the solution beyond T_m, which is a contradiction. This completes the proof.

Let V_T be a closed subspace of $C = C((-\infty, T], L^p(\Omega, X)), p \geq 2$ consisting of measurable and $B_{0,t}(dw)$–adapted processes $x = \{x(t), t \in (-\infty, T]\}$ such that $x(t) = \varphi(t), t \leq 0$. We introduce a norm in V_T as

$$||x||_{V_T} := \sup_{0 \leq t \leq T} (E||x||_t^p)^{1/p}.$$

In the following result, we work in the space V_T.

Theorem 2. *Suppose that the Assumptions (A1)–(A2) are satisfied. Then, there exists a time* $0 < T_m = T_{\max} \leq \infty$ *such that the problem (1), (2) has a unique solution. Further, if* $T_m < \infty$, *then* $\lim_{t \uparrow T_m} E|x(t)|^p = \infty$, $p \geq 2$.

4. An Example

Consider the neutral stochastic functional differential equation with finite delays r_1, r_2 and r_3 $(r > r_i \geq 0, \ i = 1, 2, 3)$:

$$\begin{aligned} d[z(t,x) + \alpha_3(t) \int_{-r_3}^{0} z(t+u,x)du] &= \alpha_1(t) \int_{-r_1}^{0} z(t+u,x)dudt \\ &+ \alpha_2(t) z(t-r_2,x) d\beta(t), \quad t > 0, \end{aligned} \tag{3}$$

where $\alpha_i = R^+ \to R^+ = [0, \infty)$, $i = 1, 2, 3$; such that $\alpha_i(t) \to 0$ as $t \to 0$, $i = 1, 2, 3; \alpha_3(t)$ is continuous, $z(s.x) = \varphi(s, x)$, $\varphi(\cdot, x) \in C_t, \varphi(s, \cdot) \in L^2[0, 1]$, $-r \leq s \leq 0$, $0 \leq x \leq 1$; and $\beta(t)$ is a standard one–dimensional Wiener process and $E||\varphi||_0^2 < \infty$.

Take $X = L^2[0,1]$, $Y = R' = (-\infty, \infty)$. Define

$$f(t, z_t) = \alpha_3(t) \int_{-r_3}^{t} z(t+u, x)du, \quad a(t, z_t) = \alpha_1(t) \int_{-r_1}^{t} z(t+u, x)du,$$

and

$$b(t, z_t) = \alpha_2(t) z(t - r_2, x).$$

Next,

$$|f(t, z_t)|^2 = \left| \alpha_3(t) \int_{-r_3}^{0} a(t+u, x)du \right|^2 \leq \alpha_3^2(t) r_3^2 ||z||_t^2, \quad \text{a.s.}$$

This shows that $f : R^+ \times C_t \to X$ and it follows that $f(t, u)$ satisfies a local Lipschitz condition with constant $\alpha_3^2(T) r_3^2$. Similarly, it can be shown that $a : R^+ \times C_t \to X$ and $b : R^+ \times C_t \to L(R', X)$. Hence equation (3) can be expressed as equation (1) with f, a and b as defined above.

Clearly,

$$|b(t, z_t^1) - b(t, z_t^2)|^2 \leq \alpha_2^2(T) ||z^1 - z^2||_t^2, \quad \text{a.s.}$$

In fact, Assumption (A2) holds as $\alpha_3(t)$ is continuous exploiting Lebesgue's Theorem. The remaining conditions can be verified similarly. Therefore, by Theorem 1, there exists a unique solution to equation (3). Note that the results from [1, 6] are not applicable in this situation. This example is motivated by Hale and Verduyn Lunel [p. 266,3].

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