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ON THE SCROLLAR INVARIANTS OF A NATURAL PENCIL ON A PARTIAL NORMALIZATION OF CERTAIN REDUCIBLE CURVES IN $\mathbb{P}^1 \times \mathbb{P}^1$

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Abstract: Here we prove that the scrollar invariants of a natural pencil on a partial normalization of certain reducible curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are as balanced as possible. We also prove that the pencil satisfies Caporaso's basic inequality for line bundles on semistable curves.

AMS Subject Classification: 14H20, 14H51
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1. Introduction

Here we prove the following two statements.

Theorem 1. *Fix integers $s \geq 2$, $a_i > 0$, $1 \leq i \leq s$, and $k_i > 0$, $1 \leq i \leq s$. Set $a := \sum_{i=1}^{s} a_i$ and $k := \sum_{i=1}^{s} k_i$. Fix an integer g such that $(a-1)k-a-k+3 \leq g \leq ak-a-k+1$. Set $e := ak-a-k+1-g$. Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a general nodal curve of type (a,k) with s irreducible components $Y_1, \dots, Y_s$ with Y_i of type (a_i, k_i) for all i. Then there exists $E \subset Sing(Y)$ such that $\sharp(E) = e$ and the partial normalization $u : X \to Y$ of Y in which we only normalize the points in E has the following properties. Let L be the pull-back of the degree k line bundle of Y associated to the first ruling of $\mathbb{P}^1 \times \mathbb{P}^1$. Then $h^0(X, L^{\otimes t}) = t+1$ for every integer $t \leq a-2$ and $h^1(X, L^{\otimes t}) = 0$ for every integer $t \geq a-1$.*

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Theorem 2. *Take the set-up of Theorem 1 with $s = 2$, $a_1 = a_2$ and $k_1 = k_2$. Then L satisfies the basic inequality of [2], i.e it is semibalanced in the sense of [4], Definition 1.1.*

The statement of Theorem 1 means that the scrollar invariants of L on the reducible curve X are as balanced as possible. For several results concerning scrollar invariants of smooth projective surfaces and their geometric interpretations, see [3], [1], papers quoted there and papers quoting them.

Proof of Theorem 1. If $e = 0$, then everything follows from the adjunction formula for the curve $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$. In the general case adjunction theory for nodal curves of the quadric surface shows that it is sufficient to find E as in the statement and such that $h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{I}_E(a-2-t, k-2)) = 0$ if $t \leq a-2$, while $h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{I}_E(a-2-t, k-2)) = 0$ if $t \geq a-1$. This is obvious, because $e \leq k-1$ and the components $Y_1, \ldots, Y_s$ are general. □

Proof of Theorem 2. We have $\deg(\omega_X) = 2ak - 2a - 2k - 2e = 8a_1k_1 - 4a_1 - 4k_1 - 2e$, $X_1 \cdot X_2 = 2a_1k_1 - e$, and $\deg(\omega_X|X_i) = 4a_1k_1 - 2a_1 - 2k_1 - e$. Hence $\deg(L)\cdot\deg(\omega_{X_i})/\deg(\omega_X) = 2k_1(4a_1k_1 - 2a_1 - 2k_1 - e)/(8a_1k_1 - 4a_1 - 2k_1 - 2e) = k_i$. Hence both inequalities of [4], equation (1.2) of Definition 1.1, are trivially satisfied. □

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