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A NOTE ON SEPARABLE POLYNOMIALS OF DEGREE 3 IN SKEW POLYNOMIAL RINGS

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Abstract: Let B be a ring with identity 1, Z the center of B, D a derivation of B, and $B[X;D]$ the skew polynomial ring such that $\alpha X = X\alpha + D(\alpha)$ for each $\alpha \in B$. Assume that $3 = 0$ and Z is a semiprime ring. Let $f = X^3 - Xa - b \in B[X;D]$ such that $fB[X;D] = B[X;D]f$. Then we prove that f is a separable polynomial in $B[X;D]$ if and only if there exits an element z in Z such that $D^2(z) - za = 1$.

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1. Introduction

Throughout this paper, B will mean a ring with identity element 1, Z the center of B, and D a derivation of B. Let $B[X;D]$ be the skew polynomial ring in which the multiplication is given by $\alpha X = X\alpha + D(\alpha)$ ($\alpha \in B$). A ring extension A/B is called separable if the A-A-homomorphism of $A \otimes_B A$ onto A defined by $a \otimes b \to ab$ splits. Let f be a monic polynomial in $B[X;D]$ such that $fB[X;D] = B[X;D]f$. Then the residue ring $B[X;D]/fB[X;D]$ is a free ring extension of B. If $B[X;D]/fB[X;D]$ is a *separable* extension of B, we call f is a *separable* polynomial in $B[X;D]$. These provide typical and essential examples of separable extensions.

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T. Nagahara gave a thorough investigation of separable polynomials of degree 2 in skew polynomial rings. K. Kishimoto, T. Nagahara, Y. Miyashita, and G. Szeto, L. Xue and the author studied extensively separable polynomials in skew polynomial rings (See References). In this paper we consider a polynomial of degree 3 in a skew polynomial ring. The purpose of this paper is to prove the following theorem. We shall prove it by making use of Miyashita's Theorem [9, Theorem 3.2].

Theorem 1. *Assume* $3 = 0$, *and let* $f = X^3 - Xa - b$ *be in* $B[X;D]$ *such that* $fB[X;D] = B[X;D]f$. *Assume that* Z *is a semiprime ring. Then* f *is a separable polymomial in* $B[X;D]$ *if and only if there exists an element* z *in* Z *such that* $D^2(z) - az = 1$.

Proof. As was verified in [1, Corollary 1.7], the hypothesis $fB[X;D] = B[X;D]f$ is equivalent to the following:

$$D(a) = D(b) = 0, \quad a \in Z \ \text{ and } \ D^3(\alpha) - D(\alpha)a = \alpha b - b\alpha \ \ (\alpha \in B).$$

Assume that f is a separable polynomial in $B[X;D]$. Then, it follows from Miyashita's theorem [9, Theorem 3.2] that there exists an element $y = X^2 d_2 + Xd_1 + d_0$ in $B[X;D]$ such that $\alpha y = y\alpha$ $(\alpha \in B)$ and $D^{*2}(y) - ya = 1$, where $D^*(\Sigma_i X^i c_i) = \Sigma_i X^i D(c_i)$. Then we obtain

$$d_2 \in Z,\ 2D(\alpha)d_2 + \alpha d_1 = d_1\alpha \ (\alpha \in B), \tag{1}$$

$$D^2(\alpha)d_2 + D(\alpha)d_1 + \alpha d_0 = d_0\alpha \ (\alpha \in B), \tag{2}$$

$$D^2(d_2) - d_2 a = 0, \tag{3}$$

$$D^2(d_1) - d_1 a = 0, \tag{4}$$

$$D^2(d_0) - d_0 a = 1. \tag{5}$$

First, we shall prove $d_2 = 0$. By the above equations, we have

$$\begin{aligned} d_2 = 1 \cdot d_2 &= \{D^2(d_0) - d_0 a\}d_2 \\ &= D^2(d_0)d_2 - d_0(ad_2) \\ &= D^2(d_0)d_2 - d_0\{D^2(d_2)\}. \end{aligned}$$

Substituting $\alpha = D(d_2) \in Z$ in (1), we have $D^2(d_2)d_2 = 0$. Hence we obtain

$$d_2^2 = D^2(d_0)d_2^2.$$

Specializing $\alpha = d_0$ in (2), we have $D^2(d_0)d_2 + D(d_0)d_1 = 0$. Then by using (1) and (2) similarly, we obtain

$$\begin{aligned} d_2^2 &= -D(d_0)d_1 d_2 = \{-D(d_0)d_2\}d_1 \\ &= \{2D(d_0)d_2\}d_1 = (d_1 d_0 - d_0 d_1)d_1 \\ &= -\{D^2(d_1)d_2 + D(d_1)d_1\}d_1 \end{aligned}$$

$$= -D^2(d_1)d_2d_1 - D(d_1)d_1^2.$$

Putting $\alpha = d_1$ in (1), we have $D(d_1)d_2 = 0$. Hence $D^2(d_1)d_2 = -D(d_1)D(d_2)$. Then we obtain

$$\begin{aligned} d_2^3 &= -D^2(d_1)d_2^2d_1 - D(d_1)d_2d_1^2 \\ &= -D^2(d_1)d_2^2d_1 = \{-D^2(d_1)d_2\}d_2d_1 \\ &= \{D(d_1)D(d_2)\}d_2d_1 \\ &= \{D(d_1)d_2\}D(d_2)d_1 = 0. \end{aligned}$$

Since Z is a semiprime ring, we have $d_2 = 0$. Thus $y = Xd_1 + d_0$. Then by $\alpha y = y\alpha$ $(\alpha \in B)$ and $D^{*2}(y) - ya = 1$ again, we obtain

$$d_1 \in Z,\ D(\alpha)d_1 + \alpha d_0 = d_0\alpha\ (\alpha \in B) \tag{6}$$

and

$$D^2(d_1) - d_1a = 0 \ \text{ and } \ D^2(d_0) - d_0a = 1. \tag{7}$$

Next, we shall show that $d_1 = 0$. By (7), we obtain

$$\begin{aligned} d_1 = 1 \cdot d_1 &= \{D^2(d_0) - d_0a\}d_1 \\ &= D^2(d_0)d_1 - d_0(ad_1) \\ &= D^2(d_0)d_1 - d_0D^2(d_1). \end{aligned}$$

Substituting $\alpha = d_0$ and $\alpha = D(d_1) \in Z$ in (6), we have

$$D(d_0)d_1 = 0 \ \text{ and } \ D^2(d_1)d_1 = 0.$$

Then we obtain $D^2(d_0)d_1 = -D(d_0)D(d_1)$, and hence

$$\begin{aligned} d_1^2 &= \{D^2(d_0)d_1 - d_0D^2(d_1)\}d_1 \\ &= -D(d_0)D(d_1)d_1 - d_0D^2(d_1)d_1 = 0. \end{aligned}$$

Since Z is a semiprime ring again, we have $d_1 = 0$. Thus $y = d_0$ is in Z.

The converse is obvious by Miyashita's Theorem [9, Theorem 3.2]. This completes the proof. □

As an immediate consequence of Theorem 1, we have the following.

Corollary 2. *Assume* $3 = 0$, *and let* $f = X^3 - Xa - b$ *be in* $B[X; D]$ *such that* $fB[X; D] = B[X; D]f$. *Assume that* Z *is an integral domain. Then* f *is a separable polymomial in* $B[X; D]$ *if and only if there exists an element* z *in* Z *such that* $D^2(z) - az = 1$.

By the similar way, we can easily show the following:

Proposition 3. *Assume* $2 = 0$, *and let* $f = X^2 - Xa - b$ *be in* $B[X; D]$ *such that* $fB[X; D] = B[X; D]f$. *Assume that* Z *is a semiprime ring. Then* f *is a separable polymomial in* $B[X; D]$ *if and only if there exists an element* z

in Z such that $D(z) - az = 1$.

Finally, we raise a question to readers.

Question. Is Theorem 1 still true for any prime number p and $f = X^p - Xa - b$?

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