International Journal of Pure and Applied Mathematics
Volume 50 No. 1 2009, 13-30

INTEGRAL EQUATION METHOD FOR THE SOLUTION OF THE DIRICHLET PROBLEM IN A PERTURBED THREE-DIMENSIONAL LAYER

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Abstract: A Dirichlet boundary value problem for the Laplace equation in a three-dimensional layer with a local perturbation of the boundary is solved by the method of boundary integral equation (IE). The unique solvability of the IE and its Fredholm property are proved. A Galerkin method of the IE numerical solution aimed at the use of parallel computations is developed and justified.

AMS Subject Classification: 35J05, 31A25, 35Q72, 35S15, 45E99
Key Words: perturbed layer, boundary integral equation, Galerkin method

1. Introduction

Boundary value problems (BVPs) in layers with perturbed boundaries were considered by many authors. Note in this respect the results of Werner et al [1, 2] where the Dirichlet BVP for the Helmholtz equation in a three-dimensional layer with a local perturbation of the boundary was considered in details.

Received: May 18, 2008

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Reduction of BVPs on manifolds with edges to boundary integral equations (IEs) is a well-known approach developed by Stephan [11], Costabel [1], Ilinskii et al [3] and others. The IE method enables one to decrease the dimension of the problem and offers a solution technique applicable in domains where the methods based on discretization fail or meet substantial difficulties.

BVPs in layers with perturbed boundaries arise, among all, in mathematical models of electromagnetics and contact mechanics. According to recent findings Shestopalov et al [10], modeling of the contact between the clichè and substrate (paper or board) in the process of flexoprinting leads to BVPs for the Lamè equation system in a 2-D or 3-D layer – the so-called boundary contact problems (BCPs). The solution to BCPs by the method of approximate decomposition employing fixed-point iterations gives arise to a sequence of auxiliary BVPs for the Laplace equation in layers having the boundary with a great amount of virtually identical disjoint local perturbations.

In this paper we develop a method of solution to the Dirichlet BVP for the Laplace equation in an 3-dimensional layer with a local perturbation Ω of the boundary based on equivalent reduction to a boundary IE over Ω. We prove the unique solvability of the IE and its Fredholm property and develop a numerical method of solution aimed at the use of parallel computations.

In view of both analytical and numerical solution, application of the boundary IE method is a urgent task, especially when problems in unbounded regions are considered. Indeed, the latter requires in particular elaboration of techniques based on the reduction and further numerical analysis of BVPs in bounded (cut) domains. Specific difficulties arise when the infinite boundary of a domain (i.e. of a layer) is perturbed by a set of many small (with respect to a characteristic dimension of the problem) inhomogeneities comparable with the mesh size. In this case, one has both to cut the domain and apply dense grids close to every inhomogeneity which leads to enormous matrix dimensions and a drastic decrease in accuracy.

The boundary IE method is to a big extent free of those drawbacks. In fact, for a BVP in a perturbed layer, the resulting IE is solved only on the inhomogeneity surface and the solution in the whole domain is determined using potentials where, again, the integration is reduced to the region occupied by inhomogeneities. What is more, the IE method proves its efficiency when IE on many disjoint regions is solved numerically using parallel computations. Elaboration of the mathematical background for the corresponding computational techniques is also an objective of this study.

2. Formulation

Consider the Dirichlet problem in a three-dimensional layer with a local perturbation of the boundary

$$\Delta u = 0, \quad x = (x_1, x_2, x_3) \in U, \tag{1}$$

$$u|_{\Omega} = \mu, \quad u|_{\partial U \setminus \Omega} = 0, \tag{2}$$

$$|u| = O\left(\frac{1}{R}\right), \quad |\nabla u| = O\left(\frac{1}{R^2}\right), \quad R := \sqrt{x_1^2 + x_2^2} \to \infty, \tag{3}$$

where

$$U = \{x : \varphi(x_1, x_2) < x_3 < 1\}$$

denotes the layer bounded by two surfaces, the plane $x_3 = 0$ and the plane $x_3 = 1$ with a local perturbation of the boundary specified by $\varphi(x_1, x_2)$. We assume that the function $x_3 = \varphi(x_1, x_2)$ is compactly supported, so that $\operatorname{supp} \varphi = \overline{\Omega}_0$, where Ω_0 is a bounded domain on the plane of variables x_1, x_2 with a piecewise smooth boundary $\partial\Omega_0$. Assume also that $0 \leq \varphi(x_1, x_2) < 1$ and $\varphi \in C^1(R^2)$ (once continuously differentiable). $\Omega = \{x : (x_1, x_2) \in \Omega_0, x_3 = \varphi(x_1, x_2)\}$ denotes the perturbed part of the boundary of U.

Function u simulates a component the field of displacements in the layer between two planes (perturbed and unperturbed) under the influence of the boundary force (pressure) specifying boundary displacements μ.

Let us formulate the BVP to be solved: it is necessary to determine the function

$$u(x_1, x_2, x_3) \in C^2(U) \bigcap C(\overline{U} \setminus \partial\Omega_0)$$

satisfying

1) the Laplace equation $\Delta u = 0$ in an unbounded domain U;

2) the boundary conditions

$$u\,|_{\Omega} = \mu(x_1, x_2, x_3)\ ,$$

where $\mu(x_1, x_2, x_3)$ is a given continuous function on the perturbed part of the boundary of Ω (and $u = 0$ everywhere on the boundary of U outside Ω);

3) the conditions at infinity (3).

3. Uniqueness and Stability

Theorem 1. *Problem (1)–(3) may have not more than one solution.*

Proof. In the layer $U_0 = \{x : 0 < x_3 < 1\}$ separate a cylinder

$$U_R = \left\{x : 0 < x_3 < 1, x_1^2 + x_2^2 < R\right\}$$

of a sufficiently large radius R.

Assuming the existence of two solutions u_1 and u_2, satisfying conditions 1)-3), we see that the difference $u_0 = u_1 - u_2$ is a solution to problem (1)–(3) with homogeneous boundary conditions (2). Applying to u_0 in U_R the maximum principle, we obtain an inequality $|u_0| \le \max\limits_{x\in\partial U_R} |u_0(x)|$. Let $x^* \in U_R$. Since condition (3) is also satisfied for function u_0, we have

$$|u_0(x^*)| \le \max_{x\in\partial U_R} |u_0(x)| \to 0, \quad R \to \infty.$$

Therefore, $u_0(x^*) = 0$ and $u_0(x) = 0$ in domain U_R, and also in a whole domain U, which proves the uniqueness of the solution of the Dirichlet BVP. □

Let us prove the stability, i.e. continuous dependence of the solution to the problem under study on the boundary data.

Recall that the problem is called *stable* (or physically well-posed) if small variation of the given data specifying the solution (that is, the boundary data) produces small variation of the solution.

To this end consider two different functions u_1 and u_2 satisfying the conditions

$$\begin{cases} \Delta u_1 = 0, x \in U \\ u_1|_\Omega = \mu_1,\, u_1|_{\partial U\setminus\Omega} = 0 \end{cases} \qquad \begin{cases} \Delta u_2 = 0, x \in U \\ u_2|_\Omega = \mu_2,\, u_2|_{\partial U\setminus\Omega} = 0 \end{cases}$$

and such that the difference of their boundary values does not exceed ε,

$$|\mu_1 - \mu_2| \le \varepsilon.$$

It is easy to verify, by applying the reasoning similar to the proof of the uniqueness theorem, that the maximum principle also holds for a spatial unbounded domain U. Therefore, the following estimate is valid

$$|u_1 - u_2| \le |\mu_1 - \mu_2| \le \varepsilon$$

which implies continuous dependence of the solution to the Dirichlet problem under study on the boundary data.

4. Reduction to a Boundary Integral Equation

We will reduce the Dirichlet problem (1)–(3) to a boundary IE using the method of Green's function.

Let us first define Green's function $G^U = G^U(x, y)$, $x, y \in U_0$ for the layer

$U_0 = \{x : 0 < x_3 < 1\}$ using the series representation Morgenröther et al [7]

$$G^U(x,y) = \frac{1}{4\pi} \sum_{j=-\infty}^{\infty} \left(\frac{1}{|x-y-2je_3|} - \frac{1}{|x-y^*+2je_3|}\right),$$

where $e_3 = (0,0,1)$ and $y^* = (y_1, y_2, -y_3)$.

Green's function satisfies the Dirichlet boundary conditions

$$G^U |_{x_3=0} = G^U |_{x_3=1} = 0$$

and vanishes therefore on the boundary of U_0 formed by two planes $x_3 = 0$ and $x_3 = 1$.

Introducing the notations

$$r_j = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3-2j)^2}$$

and

$$r'_j = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3+y_3-2j)^2},$$

we obtain an alternative formula for Green's function

$$G^U = \frac{1}{4\pi} \sum_{j=-\infty}^{\infty} \left(\frac{1}{r_j} - \frac{1}{r'_j}\right). \quad (4)$$

The method of Green's function is based on Green's formula for a domain V bounded by a piecewise smooth boundary Σ:

$$\iiint\limits_V (v\Delta u - u\Delta v)dx = \iint\limits_\Sigma \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)d\sigma, \quad (5)$$

where n is the external normal to Σ, $\frac{\partial}{\partial n} = \cos\alpha_1 \frac{\partial}{\partial x_1} + \cos\alpha_2 \frac{\partial}{\partial x_2} + \cos\alpha_3 \frac{\partial}{\partial x_3}$ is the derivative in the direction of n, and α_1, α_2, and α_3 are the angles between the external normal and axes Ox_1, Ox_2, and Ox_3, respectively. We have

$$\cos\alpha_1 = \frac{\partial\varphi}{\partial x_1} \Big/ \sqrt{1 + \left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2}, \quad \cos\alpha_2 = \frac{\partial\varphi}{\partial x_2} \Big/ \sqrt{1 + \left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2},$$

$$\cos\alpha_3 = -1 \Big/ \sqrt{1 + \left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2}.$$

Setting $v = G^U$ in (5) we obtain

$$\iiint\limits_U (G^U\Delta u - u\Delta G^U)dx = \iint\limits_\Omega \left(G^U\frac{\partial u}{\partial n} - u\frac{\partial G^U}{\partial n}\right)d\sigma\,.$$

Assuming then that u is a solution to problem (1)–(3) we see that the first term on the left-hand side vanishes and the second term has a δ-singularity at $x = y$.

Next, u on Ω equals μ which yields an integral representation for the solution

$$u(y) = \iint\limits_{\Omega} G^U \frac{\partial u}{\partial n} d\sigma - \iint\limits_{\Omega} \mu \frac{\partial G^U}{\partial n} d\sigma\,, \tag{6}$$

where μ and G^U are known functions and

$$\frac{\partial G^U}{\partial n} = \cos\alpha_1 \frac{\partial G^U}{\partial x_1} + \cos\alpha_2 \frac{\partial G^U}{\partial x_2} + \cos\alpha_3 \frac{\partial G^U}{\partial x_3}.$$

The only unknown function $\psi := \frac{\partial u}{\partial n}$ enters the first integral of representation (6), while the second integrand is a product of known functions and therefore can be calculated; we denote the second integral by

$$F = \iint\limits_{\Omega} \mu \frac{\partial G^U}{\partial n} d\sigma.$$

However, the latter integral undergoes a break on the boundary of the domain according to the properties of double-layer potentials, Vladimirov [13]. Therefore when performing a transition to the limit $x \to y \in \Omega$ in the second integral (6) the quantity $\frac{1}{2}\mu(y)$ is added,

$$\mu(y) = \iint\limits_{\Omega} G^U \psi d\sigma - F(y) + \frac{1}{2}\mu(y), \quad y \in \Omega.$$

Thus function ψ can be determined as a solution to the IE

$$\iint\limits_{\Omega} G^U \psi d\sigma = \frac{1}{2}\mu(y) + F(y), \quad y \in \Omega. \tag{7}$$

5. Integral Equation and its Properties

Write IE (7) in the operator form

$$A\psi + K\psi = f, \tag{8}$$

where

$$A\psi = \frac{1}{4\pi} \iint\limits_{\Omega} \frac{1}{|x-y|} \psi d\sigma, \tag{9}$$

$$K\psi := K_1\psi + K_2\psi, \tag{10}$$

$$K_1\psi = \frac{1}{4\pi} \iint\limits_{\Omega} \frac{1}{|x-y^*|} \psi d\sigma, \tag{11}$$

$$K_2\psi = \iint\limits_{\Omega} G_K^U \psi d\sigma, \tag{12}$$

$$f = \frac{1}{2}\mu + F, \tag{13}$$

and

$$\begin{aligned} G_K^U(x,y) &= \frac{1}{4\pi}\sum_{j=-\infty}^{-1}\left(\frac{1}{|x-y-2je_3|} - \frac{1}{|x-y^*+2je_3|}\right) \\ &+ \frac{1}{4\pi}\sum_{j=1}^{\infty}\left(\frac{1}{|x-y-2je_3|} - \frac{1}{|x-y^*+2je_3|}\right). \end{aligned} \tag{14}$$

Let M be a C^∞ 2-dimensional compact manifold without an edge in R^3. The manifold is oriented and not necessarily connected. We will consider $\overline{\Omega} \subset M$ as a submanifold with an edge of a manifold M which is not necessarily connected and has a finite number of connected components; each of these components has the dimension 2. We have that the edge $\partial\Omega$ is a piecewise smooth 1-dimensional compact manifold without edge.

We fix a finite cover $Q = \{Q_\alpha\}$ of M by coordinate vicinities and denote by $\chi_\alpha : Q_\alpha \to V_\alpha \subset R^2$ the local maps and by $\{\varphi_\alpha\}$, a partition of unity subordinated to Q. For a function $g \in C^\infty(M)$, we set $g_\alpha = \varphi_\alpha g$. For every $s \in R$, we define the Sobolev space $H^s(M)$ as a supplement of $C^\infty(M)$ with respect to the norm $\|\cdot\|_s$, where

$$\|g\|_s^2 = \sum_\alpha \|g_\alpha\|_s^2.$$

The inner product and the norm are defined for $H^s(\mathrm{R}^2)$ in a usual manner:

$$(u,v)_s = \int \langle\xi\rangle^{2s}\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,d\xi, \quad \|u\|_s^2 = (u,u)_s,$$

where

$$\langle\xi\rangle := (1 + \|\xi\|^2)^{1/2}.$$

(Here and below, we assume that the integral, for which the domain of integration is not specified, is taken over the whole space R^2.) For any other cover, partition of unity, and maps, the norms are equivalent, so that the definition of the space $H^s(M)$ is correct.

For every $s \in R$, we set, according to the notation used by Rempel et al [9]

$$H^s(\Omega) := \left\{ u\big|_\Omega : u \in H^s(M) \right\}$$

and

$$\tilde{H}^s(\overline{\Omega}) := \left\{u \in H^s(M) : \ \operatorname{supp} u \subset \overline{\Omega}\right\}.$$

The space $\tilde{H}^s(\Omega)$ may be constructed as a closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_s$. The space $\tilde{H}^s(\Omega)$ is dual to the space $H^{-s}(\Omega)$ for arbitrary s.

Consider the kernel of integral operator (9). Fix α, $Q = Q_\alpha$, and $V = V_\alpha$. Assume that the cover Q and local maps are chosen so that, in the local coordinates, the first quadratic form of $Q = Q_\alpha$ can be represented in line with Novikov et al [8] as $dl^2 = G(x_1, x_2, x_3)(\,dx_1^2 + dx_2^2 + dx_3^2)$, and $G = G_\alpha$. Let $\chi^{-1} : V \to Q$, $\mathrm{x} = \chi^{-1}(x) \in Q$, $x \in V$ be local coordinates on Q, where $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3)$. We have

$$|\mathrm{x} - y| = |\chi^{-1}(x) - \chi^{-1}(x_0)| = \Phi(x, x_0)|x - x_0|,$$

where $\Phi(x, x_0) \in C^\infty(V \times V)$ (since U is a C^∞ manifold) and $\Phi(x, x_0) > 0$ for $x, x_0 \in V$; in addition to this, $\Phi(x, x_0) = \Phi(x_0, x)$. We set

$$\Phi(x, x_0) = \Theta(x_0) + \Pi(x, x_0), \quad \Theta(x_0) := \Phi(x_0, x_0), \quad \Pi \in C^\infty(V \times V),$$

where $\Pi(x_0, x_0) = 0$ for $x_0 \in V$. Then

$$\begin{aligned}\frac{1}{|\mathrm{x} - y|} = \frac{1}{\Phi(x, x_0)|x - x_0|} &= \frac{e^{-|x-x_0|}}{\Phi(x, x_0)|x - x_0|} + \frac{1 - e^{-|x-x_0|}}{\Phi(x, x_0)|x - x_0|} \\ &= \frac{e^{-|x-x_0|}}{\Phi(x, x_0)|x - x_0|} + |x - x_0|\Phi_1(x, x_0) + \Phi_2(x, x_0) \\ &= \frac{e^{-|x-x_0|}}{\Theta(x_0)|x - x_0|} + e^{-|x-x_0|}\frac{\Pi_1(x, x_0)}{|x - x_0|} + |x - x_0|\Phi_1(x, x_0) + \Phi_2(x, x_0),\end{aligned}$$

where $\Pi_1, \Phi_1, \Phi_2 \in C^\infty(V \times V)$, and $\Pi_1(x_0, x_0) = 0$ for $x_0 \in V$.

On Q, the area differential can be represented as

$$\begin{aligned} ds &= T(x)\,dx, \quad T(x) = T(x_0) + P(x, x_0), \\ T &\in C^\infty(V), \quad P \in C^\infty(V \times V),\end{aligned}$$

where $P(x, x_0) := T(x) - T(x_0)$ and $P(x_0, x_0) = 0$ for $x_0 \in V$.

If $\Pi_0(x, x_0) \in C^\infty(V \times V)$ and $\Pi_0(x_0, x_0) = 0$ for $x_0 \in V$, then

$$\begin{aligned}&\frac{\Pi_0(x, x_0)}{|x - x_0|} \\ &= \frac{B_0(x_0)\cdot(x - x_0)}{|x - x_0|} + |x - x_0|\left(\widehat{F}_0(x_0)\frac{x - x_0}{|x - x_0|}\right)\cdot\frac{x - x_0}{|x - x_0|} + |x - x_0|\Phi_0(x, x_0),\end{aligned}$$

where vector $B_0 \in C^\infty(V)$, matrix $\widehat{F}_0 \in C^\infty(V)$, and function $\Phi_0 \in C^\infty(V \times V)$. The resulting expression takes the form

$$\frac{1}{\Phi(x,x_0)|x-x_0|}T(x) = \frac{T(x_0)e^{-|x-x_0|}}{\Theta(x_0)|x-x_0|} + \frac{B(x_0)\cdot(x-x_0)}{|x-x_0|}e^{-|x-x_0|}$$
$$+ e^{-|x-x_0|}|x-x_0|\left(\widehat{F}(x_0)\frac{x-x_0}{|x-x_0|}\right)\cdot\frac{x-x_0}{|x-x_0|}$$
$$+ |x-x_0|\tilde{\Phi}_1(x,x_0) + \tilde{\Phi}_2(x,x_0), \quad (15)$$

where $T, \Theta \in C^\infty(V)$, $T(x_0) > 0$, and $\Theta(x_0) > 0$ for $x_0 \in V$, $B, \widehat{F} \in C^\infty(V)$, and functions $\tilde{\Phi}_1, \tilde{\Phi}_2 \in C^\infty(V \times V)$.

Depending on the situation, we will consider operator A as PDOs on the manifolds M or Ω. For every coordinate vicinity $Q = Q_\alpha$, and, respectively, $V = V_\alpha$, we define the restriction of A on V by the formula

$$A_V = p_V A q_V : C_0^\infty(V) \to C^\infty(V),$$

where $q_V : C_0^\infty(V) \to C^\infty(M)$ is a natural inclusion (the zero continuation outside V) and $p_V : C^\infty(M) \to C^\infty(V)$ is the operator of restriction that transforms f to $f|_V$. If A_V is transformed into PDO, then A is PDO on the manifolds M Egorov et al [2] or Ω if to consider the restrictions of A on Ω:

$$A : C_0^\infty(\Omega) \to C^\infty(\Omega).$$

Let us define the action of A on $C_0^\infty(\Omega)$. Since it is necessary to know only the values Au at points $x \in \Omega$ (or $x \in M$), we will use the following representation for A_V:

$$A_V u = \int_V \frac{a_0(x_0)}{|x-x_0|}e^{-|x-x_0|}u(x)\,dx \quad (16)$$
$$+ \int_V e^{-|x-x_0|}\frac{B(x_0)\cdot(x-x_0)}{|x-x_0|}u(x)\,dx$$
$$+ \int_V e^{-|x-x_0|}|x-x_0|\left(\widehat{F}(x_0)\frac{x-x_0}{|x-x_0|}\right)\cdot\frac{x-x_0}{|x-x_0|}u(x)\,dx$$
$$+ \int_V \eta(|x-x_0|)|x-x_0|\Psi_1(x_0, x-x_0)u(x)\,dx$$
$$+ \int_V \eta(|x-x_0|)\Psi_2(x_0, x-x_0)u(x)\,dx. \quad (17)$$

Here,

$$a_0(x_0) := \frac{T(x_0)}{\Theta(x_0)} = \sqrt{T(x_0)},$$

$\eta(t) = 1$ for $t \le R$ and $\eta(t) = 0$ for $t \ge 2R$ is an infinitely differentiable cut function, and R is chosen so large that $\eta(|x-x_0|)\equiv 1$ for $x,x_0\in V$. In addition to this, in (17), $\Psi_i(x_0,x-x_0)=\tilde{\Phi}_i(x,x_0)$, $x,x_0\in V$ and functions $\Psi_i\in C^\infty(V\times R^2)$ are smoothly continued on R^2 with respect to the second argument. Formulas (9) and (17) obviously define one and the same operator for $x,x_0\in V$ and its definition does not depend on the choice of function η.

Each term in (17) is an integral, convolution-type operator. Let us calculate the Fourier transform of the kernels of the first three operators:

$$F\left(\frac{e^{-|x|}}{|x|}\right)=\frac{1}{\langle\xi\rangle},\quad F\left(\frac{x}{|x|}e^{-|x|}\right)=-i\frac{\xi}{\langle\xi\rangle^3},\tag{18}$$

$$F\left(e^{-|x|}|x|\left(\widehat{F}(x_0)\frac{x}{|x|}\right)\cdot\frac{x}{|x|}\right)=\frac{tr\widehat{F}(x_0)\langle\xi\rangle^2-3(\widehat{F}(x_0)\xi)\cdot\xi}{\langle\xi\rangle^5}.$$

Here, $tr\widehat{F}(x_0)$ is the trace of matrix $\widehat{F}(x_0)$. Let us denote by $b_1(x_0,\xi)$ and $b_2(x_0,\xi)$ the Fourier transforms of functions $\eta(|x|)|x|\Psi_1(x_0,x)$ and $\eta(|x|)\Psi_2(x_0,x)$ with respect to argument x:

$$b_1(x_0,\xi):=F(\eta(|x|)|x|\Psi_1(x_0,x)),\quad x_0\in V,$$
$$b_2(x_0,\xi):=F(\eta(|x|)\Psi_2(x_0,x)),\quad x_0\in V.$$

Then, (17) can be rewritten as

$$\begin{aligned}A_Vu=\int\frac{a_0(x_0)}{\langle\xi\rangle}\widehat{u}(\xi)e^{ix_0\cdot\xi}\,d\xi i\int\frac{B(x_0)\cdot\xi}{\langle\xi\rangle^3}\widehat{u}(\xi)e^{ix_0\cdot\xi}\,d\xi\\+\int\frac{tr\widehat{F}(x_0)\langle\xi\rangle^2-3(\widehat{F}(x_0)\xi)\cdot\xi}{\langle\xi\rangle^5}\widehat{u}(\xi)e^{ix_0\cdot\xi}\,d\xi\\+\int b_1(x_0,\xi)\widehat{u}(\xi)e^{ix_0\cdot\xi}\,d\xi+\int b_2(x_0,\xi)\widehat{u}(\xi)e^{ix_0\cdot\xi}\,d\xi,\end{aligned}\tag{19}$$

where $\widehat{u}(\xi)$ is the Fourier transform of function $u\in C_0^\infty(V)$ and symbols $b_1,b_2\in C^\infty(V\times R^2)$ are the Fourier transforms of finite functions. Formula (19) defines the PDO

$$A_V:C_0^\infty(V)\to C^\infty(V)$$

of the order -1 with the positively homogeneous (with respect to ξ) principal symbol $a_0(x_0)|\xi|^{-1}$.

The last term in (17) (or in (19)) produces, according to Egorov et al [2] and Rempel et al [9] an operator with an infinitely smooth kernel, so that this term belongs to $L^{-\infty}(V)$. For the function $b_1(x_0,\xi)$, we have the estimate

$$\left|\partial_{x_0}^p\partial_\xi^q b_1(x_0,\xi)\right|\le C_{K,p,q}\langle\xi\rangle^{-2+|q|}\tag{20}$$

for every compact set $K\subset V$ and arbitrary multi-indices p and q. Indeed,

$$\xi_j^2 \left|\partial_{x_0}^p \partial_\xi^q b_1(x_0,\xi)\right| = \frac{1}{2\pi}\left|\int \frac{\partial^{p+2}(\Psi_1(x_0,x)\eta(|x|)|x|)}{\partial x_0^p \partial x_j^2} e^{-ix\cdot\xi}\xi^q\,dx\right|$$
$$= \frac{|\xi|^{|q|}}{2\pi}\left|\int\left(\frac{\partial^{p+2}(\Psi_1(x_0,x)\eta(|x|))}{\partial x_0^p\partial x_j^2}|x|\right.\right.$$
$$\left.\left.+\,2\frac{\partial^{p+1}(\Psi_1(x_0,x)\eta(|x|))}{\partial x_0^p\,\partial x_j}\frac{x_j}{|x|}+\frac{\partial^p\Psi_1(x_0,x)\eta(|x|)}{\partial x_0^p}\frac{|x|^2-x_j^2}{|x|^3}\right)e^{-ix\cdot\xi}\,dx\right|$$
$$\le \tilde{C}_{K,p,q}|\xi|^{|q|} \quad (j=1,2)$$

(since the last integral converges absolutely). This estimate yields (20). From (20), it follows that $b_1 \in S^{-2}(V)$ and the fourth term in (19) is a PDO belonging to the class $L^{-2}(V)$ Rempel et al [9].

Thus, for the PDO A_V, we have the representation

$$\begin{aligned} A_V u \;\; &\equiv \;\; A_V^0 u + B_V u = \int a_V(x_0,\xi)\widehat{u}(\xi)e^{ix_0\cdot\xi}\,d\xi \\ &= a_0(x_0)\int \frac{1}{\langle\xi\rangle}\widehat{u}(\xi)e^{ix_0\cdot\xi}\,d\xi + \int b_V(x_0,\xi)\widehat{u}(\xi)e^{ix_0\cdot\xi}\,d\xi, \qquad (21) \\ a_V(x_0,\xi) &:= a_0(x_0)\langle\xi\rangle^{-1} + b_V(x_0,\xi), \end{aligned}$$

where the symbol $b_V \in S^{-2}(V)$ and $B_V \in L^{-2}(V)$.

Consider the operator $\tilde{A}$ defined by the formula

$$\tilde{A}u = \sum_\alpha \psi_\alpha A_\alpha \varphi_\alpha u, \qquad (22)$$

or

$$\tilde{A}u \equiv \tilde{A}^0 u + \tilde{B}u = \sum_\alpha \psi_\alpha A_\alpha^0 \varphi_\alpha u + \sum_\alpha \psi_\alpha B_\alpha \varphi_\alpha u, \qquad (23)$$

where $A_\alpha \equiv A_V$, $A_\alpha^0 \equiv A_V^0$, $B_\alpha \equiv B_V$ for $V = V_\alpha$, $1 = \sum\limits_\alpha \varphi_\alpha$, and functions ψ_α are such that $\operatorname{supp}\psi_\alpha \subset V_\alpha$ and $\psi_\alpha\varphi_\alpha \equiv \varphi_\alpha$. The operator $\tilde{A} : C_0^\infty(\Omega) \to C^\infty(\Omega)$.

Since the kernel of A in (9) has a singularity only at $x = y$, operator A differs from $\tilde{A}$ by an operator with an infinitely smooth kernel:

$$A = \tilde{A} + \tilde{K}, \quad \tilde{K} \in L^{-\infty}(\Omega). \qquad (24)$$

The PDO with the symbol $\langle\xi\rangle^{-1}$ performs an isomorphic mapping of $H^{-1/2}(M)$ on $H^{1/2}(M)$ Mishchenko [5]. Since $a_0(x_0) > 0$ in every coordinate vicinity, $C_0^\infty(\Omega)$ is dense in $\tilde{H}^{-1/2}(\overline{\Omega})$, and $\tilde{H}^{-1/2}(\overline{\Omega})$ is antidual with respect to $H^{1/2}(\Omega)$, operator $\tilde{A}^0$ can be continued up to a bounded, continuously invertible operator

$$\tilde{A}^0 : \tilde{H}^{-1/2}(\overline{\Omega}) \to H^{1/2}(\Omega).$$

Then, by virtue of (23) and (24),

$$A : \tilde{H}^{-1/2}(\overline{\Omega}) \to H^{1/2}(\Omega)$$

is a bounded Fredholm operator with the zero index.

Look for the solution to equation (7) in the Sobolev space $\psi \in \tilde{H}^{-1/2}(\overline{\Omega})$; the right hand side $f \in H^{1/2}(\Omega)$; and the integral operators A and K are defined appropriately

$$A : \tilde{H}^{-1/2}(\overline{\Omega}) \to H^{1/2}(\Omega),$$
$$K : \tilde{H}^{-1/2}(\overline{\Omega}) \to H^{1/2}(\Omega).$$

We have proved that operator A is a bounded and Fredholm operator (with the zero index) in the chosen spaces.

Since the smooth term $G_K^U(x, y)$ of Green's function and its derivatives of arbitrary order with respect to variables x and y are continuous in $\overline{\Omega} \times \overline{\Omega}$ (and has no singularity at $x = y$), i.e. $G_K^U \in C^\infty(\overline{\Omega} \times \overline{\Omega})$, according to Morgenröther et al [7] and Egorov et al [2], p. 85, operator K_2 is compact in these spaces.

Consider integral operator $K_1 : \tilde{H}^{-1/2}(\overline{\Omega}) \to H^{1/2}(\Omega)$. For the kernel of the integral operator K_1 we have $\frac{1}{|x-y^*|} \in C^\infty(\overline{\Omega} \times \Omega)$ (and $\frac{1}{|x-y^*|} \in C^\infty(\Omega \times \overline{\Omega})$) because the kernel has singularity only on the boundary $\partial\Omega$ for $x = y^* \in \partial\Omega$. It can be shown that K_1 is also a compact operator.

Thus $A + K : \tilde{H}^{-1/2}(\overline{\Omega}) \to H^{1/2}(\Omega)$ is a Fredholm operator (with the zero index) Taylor [12]. According to Theorem 1 the BVP has not more than one solution, which implies that the homogeneous equation $A\psi + K\psi = 0$ (with $f = 0$) has only the trivial solution. Indeed if this homogeneous equation would have a nontrivial solution ψ, then, substituting $\frac{\partial u}{\partial n} = \psi$ and $\mu = 0$ in (6), we obtain a nontrivial solution u to BVP (1)–(3), which contradicts the statement of Theorem 1. In addition, the operator $A+K$ has a bounded inverse $(A + K)^{-1} : H^{1/2}(\Omega) \to \tilde{H}^{-1/2}(\overline{\Omega})$.

We have proved the following

Theorem 2. *The solution $\psi \in \tilde{H}^{-1/2}(\overline{\Omega})$ to IE $A\psi + K\psi = f$ (and to equation (7)) exists and is unique for every right-hand side $f \in H^{1/2}(\Omega)$.*

Consider IE (7), to which the problem (1)–(3) has been reduced. Transform the surface integral over Ω to a double integral over Ω_0:

$$\iint\limits_{\Omega_0} \tilde{G}^U(x, y)\tilde{\psi}(x)d\,x_1 d\,x_2 = \tilde{F}(y), \quad y = (y_1, y_2) \in \Omega_0, \tag{25}$$

$$\tilde{F}(y_1, y_2) = 1/2\mu(y_1, y_2, \varphi(y_1, y_2)) + F(y_1, y_2, \varphi(y_1, y_2)), \tag{26}$$

$$\tilde{G}^U(x_1, x_2; y_1, y_2) = G^U(x_1, x_2, \varphi(x_1, x_2); y_1, y_2, \varphi(y_1, y_2)), \tag{27}$$

$$\tilde{\psi}(x_1, x_2) = \psi(x_1, x_2, \varphi(x_1, x_2))\sqrt{1 + (\frac{\partial\varphi}{\partial x_1})^2 + (\frac{\partial\varphi}{\partial x_2})^2}. \tag{28}$$

IE in the form (25)–(28) will be solved numerically by the Galerkin method described in Section 6.

6. Galerkin Method

Consider the scheme of the Galerkin method for the solution of IE (8)according to Kress [4]

$$\langle (A + K)\psi_N, u_k\rangle = \langle f, u_k\rangle, \quad k = 1, \ldots, N. \tag{29}$$

Here $\psi_N \in H_N$ is the approximate solution, $u_k \in H_N$ are basis functions, and $H_N \subset H = \tilde{H}^{-1/2}$ finite-dimensional spaces. Brackets $\langle\cdot,\cdot\rangle$ denote the duality relation on the pair of dual spaces H' and H, where $H' = H^{1/2}(\Omega)$ and $H = \tilde{H}^{-1/2}(\overline{\Omega})$, with respect to the bilinear form $\langle\psi, f\rangle = \iint\limits_{\Omega} \psi f \, d\,\sigma$.

Stephan [11] and Kress [4] proved that the Galerkin method (29) converges if the following conditions of approximation are fulfilled

$$\inf_{u_N \in H_N} \|u_N - \psi\| \to 0, \quad N \to \infty \tag{30}$$

for every $\psi \in H$.

We will solve IE (25) numerically by the Galerkin method. Let $\Omega_0 = \Pi$, where $\Pi := \{x : 0 < x_1 < a_1, 0 < x_2 < a_2\}$ is a rectangle. Choose in Π a uniform rectangular grid with the nodes (x_1^i, x_2^j), $x_1^i = ih_1$, $x_2^j = jh_2$, where $h_1 = \frac{a_1}{N_1}$, $h_2 = \frac{a_2}{N_2}$, $N_1 \geq 1$, $N_2 \geq 1$, $i = 0, \ldots, N_1$, $j = 0, \ldots, N_2$. Define basis functions of the Galerkin method

$$u_{ij} = (\chi(x_1 - x_1^i) - \chi(x_1 - x_1^{i+1}))(\chi(x_2 - x_2^j) - \chi(x_2 - x_2^{j+1})),$$

where $\chi(t) = 0$ for $t < 0$ and $\chi(t) = 1$ for $t \geq 0$. Thus every basis function has a compact support $\Pi_{ij} := \{x : x_1^i \leq x_1 \leq x_1^{i+1}, x_2^j \leq x_2 \leq x_2^{j+1}\}$.

Applying the Galerkin method we obtain a linear equation system

$$\sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} a_{ijkl} c_{ij} = f_{kl}, k = 0, \ldots, N_1 - 1, l = 0, \ldots, N_2 - 1, \tag{31}$$

where entries a_{ijkl} and components of the right-hand side f_{kl} are calculated as multiple:

$$a_{ijkl} = \iint\limits_{\Pi_{ij}} \iint\limits_{\Pi_{kl}} \tilde{G}^U(x_1, x_2; y_1, y_2) d\,x_1 d\,x_2 d\,y_1 d\,y_2, \tag{32}$$

$$f_{kl} = \iint\limits_{\Pi_{kl}} \tilde{F}(y_1, y_2) d\,y_1 d\,y_2. \tag{33}$$

Approximate solution $\tilde{\psi}_{N_1,N_2}$ is represented as

$$\tilde{\psi}_{N_1,N_2} = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} c_{ij} u_{ij}. \tag{34}$$

A general advantage of the proposed method in comparison with the existing techniques is that a 2D equation is solved in the domain occupied by the inhomogeneity rather than a 3D problem in the unbounded domain.

If the perturbation of the boundary is sufficiently complicated, computational costs may be substantial, so that it would not be possible to solve the problem numerically with a reasonable accuracy and time even using modern powerful computers.

The only chance to solve such problems efficiently is application of parallel computations using multiprocessor systems (clusters), or supercomputers. However, they require creation of the specific problem-oriented algorithms of parallel computations.

We propose the following algorithms of parallel computations for the solution of the problem under study based on the above implementation of the Galerkin method.

1. Integrals forming the entries of the matrix of the linear algebraic system are calculated in parallel using several (M) different processors. Since the data exchange between the processors in the course of computations is not necessary, the rate of computations increases by a factor of M.

2. If the given function μ is a superposition of a big number of functions with nonoverlapping supports, then the problem can be solved separately for every such function on a separate processor and the general solution can be obtained then as a superposition of particular solutions of separate problems. An advantage here is in the following: Denote by k (e.g. equal to 10^{-m}, $m \geq 3$) the ratio of the diameter of the support of a separate small subdomain to the diameter of the support of the whole domain occupied by the inhomogeneity. Then the ratio of the respective computation times is proportional to k.

3. The linear algebraic system of the Galerkin method can be also solved using algorithms of parallel computations and clusters if we apply iteration techniques (for example, a parallel version of the method of conjugate gradients). In fact, the most time-consuming operation of the matrix–vector multiplication occurring in algorithms based on iterations can be performed in parallel on different processors.

7. Conclusion

The boundary IE method has been applied to both mathematical and numerical analysis of a Dirichlet BVP for the Laplace equation in a three-dimensional layer with a locally perturbed boundary. The unique solvability of the IE and its Fredholm property have been proved. A Galerkin method of the IE numerical solution has been developed and justified. It has been shown how one can implement the method using, in particular, parallel computations.

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Appendix. Green's Function and its Properties

Let us show that series (4) as well as series obtained as a result of termwise differentiation of (4) converge absolutely and uniformly in the layer $0 < x_3 < 1$.

Prove first uniform convergence of series (4). To this end consider its j-th term for $j > 0$ (the case $j < 0$ can be considered analogously)

$$a_j = \frac{1}{r_j} - \frac{1}{r'_j} = \frac{4y_3(x_3 - 2j)}{r_j r'_j (r_j + r'_j)};$$

the following inequality is valid

$$|a_j| \le \frac{8jy_3}{(2j-1)(2j-1-y_3)(4j-3)},$$

because $r_j \geq (2j-1)$ and $r'_j \geq (2j-1-y_3)$. The estimate shows that the series $\sum_{j=1}^{\infty} a_j$ (and analogously $\sum_{j=-\infty}^{-1} a_j$) converge absolutely and uniformly in the layer with respect to the variables x_1, x_2, x_3 and y_1, y_2 and also with respect to y_3 if one removes the term with $j = 1$.

Let us show now that series obtained as a result of termwise differentiation of (4) converge absolutely and uniformly in the domain $0 < x_3 < 1$.

To this end, estimate the derivatives for $j > 0$

$$\left|\frac{\partial}{\partial x_3}\left(\frac{1}{r_j}\right)\right| = \left|-\frac{1}{r_j^2}\frac{x_3 - y_3 - 2j}{r_j}\right| < \frac{1}{r_j^2},$$

because $\left|\frac{x_3-y_3-2j}{r_j}\right| < 1$, and

$$\left|\frac{\partial}{\partial x_3}\left(\frac{1}{r'_j}\right)\right| = \left|-\frac{1}{r'^2_j}\frac{x_3 + y_3 - 2j}{r'_j}\right| < \frac{1}{r'^2_j},$$

because $\left|\frac{x_3+y_3-2j}{r'_j}\right| < 1$; also

$$\frac{\partial^2}{\partial x_3^2}\left(\frac{1}{r_j}\right) = \frac{3(x_3 - y_3 - 2j)^2}{r_j^5} - \frac{1}{r_j^3}$$

and

$$\left|\frac{\partial^2}{\partial x_3}\left(\frac{1}{r_j}\right)\right| < \frac{4}{r_j^3}.$$

Inequalities $r_j > j$ and $r'_j > j$ yield

$$\left|\frac{\partial a_j}{\partial x_3}\right| < \frac{2}{j^2}, \quad \left|\frac{\partial^2 a_j}{\partial x_3^2}\right| < \frac{8}{j^3},$$

which implies absolute and uniform convergence of series $\sum_{j=1}^{\infty} \frac{\partial a_j}{\partial x_3}$ and $\sum_{j=1}^{\infty} \frac{\partial^2 a_j}{\partial x_3^2}$. A similar result holds for the series

$$\sum_{j=1}^{\infty} \frac{\partial a_j}{\partial x_1}, \quad \sum_{j=1}^{\infty} \frac{\partial a_j}{\partial x_2}, \quad \sum_{j=1}^{\infty} \frac{\partial^2 a_j}{\partial x_1^2}, \quad \sum_{j=1}^{\infty} \frac{\partial^2 a_j}{\partial x_2^2}.$$

The same result can be obtained for the series

$$\sum_{j=-\infty}^{-1} \frac{\partial a_j}{\partial x_3}, \sum_{j=-\infty}^{-1} \frac{\partial^2 a_j}{\partial x_3^2}, \sum_{j=-\infty}^{-1} \frac{\partial a_j}{\partial x_1}, \quad \sum_{j=-\infty}^{-1} \frac{\partial a_j}{\partial x_2}, \quad \sum_{j=-\infty}^{-1} \frac{\partial^2 a_j}{\partial x_1^2}, \quad \sum_{j=-\infty}^{-1} \frac{\partial^2 a_j}{\partial x_2^2}.$$

Thus series (4) without the term $\frac{1}{r_0}$ satisfies (termwise) the Laplace equation

$$\Delta G^U = 0$$

everywhere in the layer $0 < x_3 < 1$. The first (extracted) term $\frac{1}{r_0}$ gives the required singularity.