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ON THE L-ORDER AND L-TYPE OF DIFFERENTIAL POLYNOMIALS

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Abstract: In the paper we study the relationship between the L-order (L-type) of a meromorphic function and that of a differential polynomial generated by it.

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1. Introduction, Definitions and Notations

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_{0j}, n_{1j}, \ldots, n_{kj}$ $(k \geq 1)$ be non-negative integers such that for each j, $\sum\limits_{i=0}^{k} n_{ij} \geq 1$. We call $M_j[f] = A_j (f)^{n_{0j}} (f^{(1)})^{n_{1j}} \ldots (f^{(k)})^{n_{kj}}$ where

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$T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The numbers $\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$ and $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$ are called respectively the degree and the weight of $M_j[f]$, see [2], [5]. The expression $P[f] = \sum_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max_{1<j<s} \gamma_{M_j}$ and $\Gamma_P = \max_{1<j<s} \Gamma_{M_j}$ are called respectively the degree and weight of $P[f]$, see $\{[2], [5]\}$. Also we call the numbers $\underline{\gamma_P} = \min_{1<j<s} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma_P} = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f, i.e. for which $n_{0j} = 0$ for $j = 1, 2, ..., s$. We consider only those $P[f], P_0[f]$ singularities of whose individual terms do not cancel each other.

The following definitions are well known.

Definition 1. The order ρ_f and lower order λ_f of a meromorphic function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If f is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Definition 2. The hyper order $\bar{\rho}_f$ and hyper lower order $\bar{\lambda}_f$ of a meromorphic function f is defined as

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r, f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r, f)}{\log r}.$$

If f is entire then one can easily verify that

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r, f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r, f)}{\log r}.$$

Definition 3. The type σ_f of a meromorphic function f is defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

When f is entire, then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Somasundaram and Thamizharasi [6] introduced the notion of $L-$order and $L-$type for entire functions where $L = L(r)$ is a positive continuous function increasing slowly, i.e. $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Their definitions are as follows:

Definition 4. (see [6]) The L-order ρ_f^L and the L-lower order λ_f^L of an entire function f are defined as follows:

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log [rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log [rL(r)]}.$$

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log [rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log [rL(r)]}.$$

Definition 5. (see [6]) The $L-$type σ_f^L of an entire function f with L-order ρ_f^L is defined as

$$\sigma_f^L = \limsup_{r\to\infty} \frac{\log M(r,f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

For meromorphic f, the $L-$type σ_f^L becomes

$$\sigma_f^L = \limsup_{r\to\infty} \frac{T(r,f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

Similarly one can define the $L-$hyper order and $L-$lower hyper order of entire and meromorphic f. The more generalised concept of L-order and L-type of entire and meromorphic functions are respectively L^*-order and L^*-type. Their definitions are as follows:

Definition 6. The L^*-order, L^*-lower order and L^*-type of a meromorphic function f are respectively defined by

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log \left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log \left[re^{L(r)}\right]}$$

and

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*}=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log\left[re^{L(r)}\right]},\quad \lambda_f^{L^*}=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log\left[re^{L(r)}\right]}$$

and

$$\sigma_f^{L^*}=\limsup_{r\to\infty}\frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}\text{ where }0<\rho_f^{L^*}<\infty.$$

We also require the following definitions.

Definition 7. The quantity $\Theta(a;f)$ of a meromorphic function f is defined as follows

$$\Theta(a;f)=1-\limsup_{r\to\infty}\frac{\bar{N}(r,a;f)}{T(r,f)}.$$

Definition 8. (see [4]) For $a\in\mathbb{C}\cup\{\infty\}$, let $n_p(r,a;f)$ denotes the number of zeros of $f-a$ in $|z|\le r$, where a zero of multiplicity $<p$ is counted according to its multiplicity and a zero of multiplicity $\ge p$ is counted exactly p times; and $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way. We define

$$\delta_p(a;f)=1-\limsup_{r\to\infty}\frac{N_p(r,a;p)}{T(r,f)}.$$

Definition 9. (see [1]) $P[f]$ is said to be admissible if: $(i)P[f]$ is homogeneous, or $(ii)P[f]$ is non homogeneous and $m(r,f)=S(r,f)$.

In the paper we establish the relationship between the $L-$order of $P_0[f]$ and f. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [7] and [3].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. (see [1]) *Let $P_0[f]$ be admissible. If f is of finite order or of non zero lower order and $\sum_{a\neq\infty}\Theta(a;f)=2$ then*

$$\lim_{r\to\infty}\frac{T(r,P_0[f])}{T(r,f)}=\Gamma_{P_0}.$$

Lemma 2. (see [1]) *Let f be either of finite order or of non-zero lower order such that* $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$. *Then for homogeneous* $P_0[f]$,

$$\lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r, f)} = \gamma_{P_0.}$$

Lemma 3. (see [4]) *Let f be a meromorphic function of finite order or of non zero lower order such that* $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$. *Then for every homogeneous* $P_0[f]$,

$$\lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r, f)} = \gamma_{P_0.}$$

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *If f be a meromorphic function of finite order or of non zero lower order and* $\sum_{a\neq\infty} \Theta(a; f) = 2$, *then the $L-$order of $P_0[f]$ are same as those of f and the $L-$type of $P_0[f]$ is Γ_{P_0} times that of f.*

Proof. By Lemma 1, $\lim_{r\to\infty} \frac{\log T(r,P_0[f])}{\log T(r,f)}$ exists and is equal to 1. So

$$\rho^L_{P_0[f]} = \limsup_{r\to\infty} \frac{\log T(r, P_0[f])}{\log[rL(r)]}$$

$$= \limsup_{r\to\infty} \{\frac{\log T(r, f)}{\log[rL(r)]} . \frac{\log T(r, P_0[f])}{\log T(r, f)}\}$$

$$= \limsup_{r\to\infty} \frac{\log T(r, f)}{\log[rL(r)]} . \lim_{r\to\infty} \frac{\log T(r, P_0[f])}{\log T(r, f)}$$

$$= \rho^L_f .1 = \rho^L_f .$$

Again

$$\sigma^L_{P_0[f]} = \limsup_{r\to\infty} \frac{T(r, P_0[f])}{[rL(r)]^{\rho^L_{P_0[f]}}}$$

$$= \limsup_{r\to\infty} \frac{T(r, f)}{[rL(r)]^{\rho^L_f}} . \lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r, f)}$$

$$= \sigma^L_{f.} . \Gamma_{P_0} .$$

This proves the theorem. □

Theorem 2. *Let f be a meromorphic function of finite order or of non zero lower order. If $\sum_{a\neq\infty}\Theta(a,f)=2$, then the $L-$lower order of $P_0[f]$ and that of f are equal.*

We omit the proof of Theorem 2 because it can be carried out in the line of Theorem 1.

Remark 1. The conclusions of Theorem 1 and Theorem 2 can also be drawn under the hypothesis $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ instead of $\sum_{a\neq\infty}\Theta(a;f)=2$.

Theorem 3. *Let f be a meromorphic function of finite order or of non zero lower order. If $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ then the $L-$hyper order of $P_0[f]$ are same as that of f.*

Proof. Let $\overline{\rho}^L_{P_0[f]}$ and $\overline{\rho}^L_f$ be the $L-$hyper orders of $P_0[f]$ and f respectively. By Lemma 2, $\lim_{r\to\infty}\frac{\log^{[2]}T(r,P_0[f])}{\log^{[2]}T(r,f)}$ exists and is equal to 1. Thus we get

$$\overline{\rho}^L_{P_0[f]}=\limsup_{r\to\infty}\frac{\log^{[2]}T(r,P_0[f])}{\log[rL(r)]}$$

$$=\limsup_{r\to\infty}\{\frac{\log^{[2]}T(r,f)}{\log[rL(r)]}.\frac{\log^{[2]}T(r,P_0[f])}{\log^{[2]}T(r,f)}\}$$

$$=\limsup_{r\to\infty}\frac{\log^{[2]}T(r,f)}{\log[rL(r)]}.\lim_{r\to\infty}\frac{\log^{[2]}T(r,P_0[f])}{\log^{[2]}T(r,f)}$$

$$=\overline{\rho}^L_f.1=\overline{\rho}^L_f.$$

Thus the theorem is established. □

In the line of Theorem 3 we may state the following theorem without proof.

Theorem 4. *Let f be a meromorphic function of finite order or of non zero lower order. If $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ then the $L-$hyper lower orders of $P_0[f]$ and f are same.*

Remark 2. The conclusions of Theorem 3 and Theorem 4 can also be deduced under the hypothesis $\sum_{a\neq\infty}\Theta(a;f)=2$ or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ instead of $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$.

In the following theorems we establish the relationship between the L^*-order (L^*-type) and L^*-hyper order of $P_0[f]$ and f.

Theorem 5. *If f be a meromorphic function of finite order or of non zero lower order and $\sum_{a\neq\infty}\Theta(a;f)=2$, then the L^*-order of $P_0[f]$ is same as that of f and the L^*-type of $P_0[f]$ is Γ_{P_0} times that of f.*

We omit the proof of Theorem 5 because it can be carried out in the line of Theorem 1.

Theorem 6. *Let f be a meromorphic function of finite order or of non zero lower order. If $\sum_{a\neq\infty}\Theta(a,f)=2$ then the L^*-lower order of $P_0[f]$ and that of f are equal.*

The proof is omitted.

Remark 3. The conclusions of Theorem 5 and Theorem 6 can also be drawn under the hypothesis $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ instead of $\sum_{a\neq\infty}\Theta(a;f)=2$.

Similarly one can prove the following two theorems in view of Lemma 3 and in the line of Theorem 3.

Theorem 7. *Let f be a meromorphic function of finite order or of non zero lower order. If $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ then the L^*-hyper order of $P_0[f]$ is same as that of f.*

Theorem 8. *Let f be a meromorphic function of finite order or of non zero lower order. If $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ then the L^*-hyper lower orders of $P_0[f]$ and f are same.*

Remark 4. The conclusions of Theorem 7 and Theorem 8 can also be deduced under the hypothesis $\sum_{a\neq\infty}\Theta(a;f)=2$ or $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ instead of $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$.

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