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A NORMAL FORM OF A HOLONOMIC q-DIFFERENCE SYSTEM AND ITS APPLICATION TO BC_1 TYPE

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Abstract: We obtain a normal form of holonomic q-difference equations which is locally holomorphic at the origin and apply it to the q-difference equation satisfied by Jackson integrals of BC_1 type.

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1. Introduction

Throughout this paper, we assume $0 < q < 1$ and denote the *q-shifted factorial* for all integers N by $(x;q)_\infty := \prod_{i=0}^{\infty}(1-q^i x)$ and $(x;q)_N := (x;q)_\infty/(q^N x;q)_\infty$.

Let $x_1, \dots, x_m$ ($m = 2s+2$) be even number of independent variables such that $x_k \in \mathbf{C}^*$. Let T_{x_k} be the difference operator corresponding to the q-shift : $x_k \to q x_k$. We denote by ϵ_k the k-th elementary symmetric polynomials in $x_1, x_2, \dots, x_m$.

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Let $\Phi(t)$ the q-multiplicative function of BC_1 type

$$\Phi(t) = t^{m/2-\delta} \prod_{k=1}^{m} \frac{(qx_k^{-1}t;q)_\infty}{(x_k t;q)_\infty}\,,$$

where we put $q^\delta = x_1 x_2 \cdots x_m$ (see [4], [7], [10], etc.). We take a skew-symmetric Laurent polynomial $\varphi(t)$ in t such that $\varphi(t) = -\varphi(1/t)$ and a point $\xi \in \mathbf{C}^*$. The Jackson integral over the lattice point $\langle\xi\rangle = \{\xi q^\nu,\ \nu = 0, \pm 1, \pm 2, \ldots\}$ in $\mathbf{C}^*$ is defined as the sum

$$\int_{\langle\xi\rangle} \Phi(t)\varphi(t)\frac{d_q t}{t} = (1-q)\sum_{\nu=-\infty}^{\infty} \Phi(\xi q^\nu)\varphi(\xi q^\nu)$$

provided it is summable. This will be denoted by $\langle\varphi, \xi\rangle$.

We consider a holonomic system of linear q-difference equations relative to $y_1, \ldots, y_s$ as functions of $x_1, \ldots, x_m$

$$T_{x_k} y_j = y_{j-1} - (x_k + 1/x_k) y_j + y_{j+1} \quad (1 \le j \le s-1)$$

$$T_{x_k} y_s = y_{s-1} - (x_k + 1/x_k) y_s + \sum_{j=1}^{s} (-1)^{s-j} \frac{\epsilon_{s-j+1} - \epsilon_{s+j+1}}{1-\epsilon_m} y_j\,, \tag{1}$$

where we put $y_0 = 0$.

By cohomological argument, we have proved in [4], [7], [8] that if we take $\varphi_k(t) = t^k - t^{-k}$, $(1 \le k \le s)$, the Jackson integrals $y_k = \langle\varphi_k, \xi\rangle$ satisfy the q-difference equations (1).

One can represent (1) in matrix form relative to the vector function $\mathbf{y} = (y_1, \ldots, y_s)$ as follows

$$T_{x_k}\mathbf{y} = \mathbf{y} A_k\,, \tag{2}$$

$$A_k = -(x_k + 1/x_k) I + A(\emptyset) + \sum_{j=1}^{s} (-1)^{s-j} \frac{\epsilon_{s-j+1} - \epsilon_{s+j+1}}{1-\epsilon_m} E_{js}\,,$$

$$A(\emptyset) = \sum_{j=1}^{s-1} E_{j,j+1} + E_{j+1,j}\,,$$

where E_{ij} denotes the matrix with only the non-zero (ij)-component equal to 1.

The author investigated in [2] a normal form of a holonomic system of q-difference equations at a boundary of regular singularity. But the equations (2) have not only an irregular singularity but also have the principal part equal to the identity matrix. So the result therein is not applicable.

The purpose of this note is to prove Theorem 2 and the following theorem as an application.

Theorem 1. *There exists the unique matrix $U = U(x)$ for $x = (x_1, \dots, x_m)$ which is holomorphic in a neighbourhood of the origin and having the expansion symmetric with respect to $x_1, \dots, x_m$*

$$U = I + \sum_{\nu=2}^{\infty} U^{(\nu)}\,, \tag{3}$$

such that

$$A_k = U(-1/x_k I + A(\emptyset))\{T_{x_k} U\}^{-1}\,. \tag{4}$$

Here $U^{(\nu)}$ denotes the homogeneous part consisting of the terms of degree ν.

One can see that $U^{(2)}, U^{(3)}$ have the expressions

$$\begin{aligned} U^{(2)} &= \sum_{j,k=1}^{m} U(jk)x_j x_k \quad (U(jk) = U(kj))\,, \\ U^{(3)} &= \sum_{j,k,l=1}^{m} U(jkl)x_j x_k x_l \quad (U(jkl) = U(kjl) = U(klj))\,, \end{aligned}$$

such that

$$U(jj) = \frac{1}{1-q^2}\sum_{l=1}^{s-1} E_{ll}, \;\; U(jk) = -\frac{1}{2(1-q)}E_{ss} \quad (j \neq k)\,,$$

and that

$$\begin{aligned} U(jjj) &= \frac{\{\sum_{h=1}^{s-2}(1-q^2)(E_{h,h+1} + E_{h+1,h}) + E_{s-1,s} - q^2 E_{s,s-1}\}}{(1-q^2)(1-q^3)}\,, \\ U(jjk) &= \frac{1}{3(1-q)(1-q^2)}\{E_{s-1,s} - E_{s,s-1}\}\,, \\ U(jkl) &= \frac{1}{6(1-q)^2}\{(2-q)E_{s-1,s} - E_{s,s-1}\}\,, \end{aligned}$$

for different j, k, l.

We denote by $\theta(a;q) = (a;q)_\infty (q/a;q)_\infty (q;q)_\infty$ the Jacobi theta function. Put

$$z_k = \frac{y_k}{\prod_{j=1}^{m}\theta(x_j;q)}\,.$$

Then the equations (2) can be rewritten as

$$T_{x_k}\mathbf{z} = \mathbf{z}B_k(x) \tag{5}$$

relative to the vector function $\mathbf{z} = (z_1, \dots, z_m)$, where each of $B_k(x) = -x_k A_k(x)$ is holomorphic in a neighborhood of the origin. One can see that (5) is a holonomic system in the following sense. For arbitrary j, k the compatibility condition holds:

$$B_j \cdot T_{x_j} B_k = B_k \cdot T_{x_k} B_j \,. \tag{6}$$

2. Proof of Theorem

We consider holonomic q-difference equations in a slightly more general form than in Section 1.

Let B_k be square matrices of degree s which are holomorphic at the origin $x = 0$, having the power series expansion

$$B_k = I + \sum_{r=1}^{\infty} \sum_{J,|J|=r} B_k(J) x_J \,,$$

where the coefficients $B_k(J) = B_k(j_1, \ldots, j_r)$ for the monomials $x_J = x_{j_1} \cdots x_{j_r}$ satisfy

$$B_k(j_1, \ldots, j_r) = B_k(j_{\sigma(1)}, \ldots, j_{\sigma(r)})$$

for any permutation σ. We assume that B_k *satisfies the compatibility condition* (6) *i.e., that the equations* (5) *are holonomic.* Then

$$B_j(k) = 0 \quad (j \neq k), \; B_j(k, l) = 0 \quad (j \neq k, l) \tag{7}$$

and

$$B_j(j, k) - B_k(j, k) = \frac{1}{2}[B_k(k), B_j(j)] \,.$$

Moreover the following lemma holds.

Lemma 2.1.

$$B_k(x) - I \equiv 0 \;\; mod \; (x_k) \,, \tag{8}$$

i.e.

$$B_k(J) = 0 \tag{9}$$

provided $|J| \geq 1$ *and* $J \cap \{k\} = \emptyset$.

Proof. We prove (9) by induction on $r = |J|$. When $r = 1$ this reduces to (7). Suppose that the lemma is true for $r < r^*$. We must prove it when $|J| = r^*$. Suppose $j \in J$. Let $\nu_j > 0$ be the number of the index j contained in J. Compare the coefficients of x_J in both sides of (6). Since $B_k(K) = 0$ for $K \subsetneq J$, we have

$$B_j(J) + B_k(J) q^{\nu_j} = B_k(J) + B_j(J)$$

whence $B_k(J) = 0$ because $q^{\nu_j} \neq 1$. The lemma is now true. □

Under this circumstance, the following holds.

Theorem 2. *The holonomic q-difference equations*

$$T_{x_j}\mathbf{z} = \mathbf{z}B_j \quad (1 \leq j \leq m) \tag{10}$$

has the unique fundamental solution matrix Z which is holomorphic in a neighborhood of the origin, satisfying $Z = I$ at the origin, so that Z has the expansion

$$Z = I + \sum_{r=1}^{\infty} \sum_{J,|J|=r} Z(J)x_J \,.$$

Before proving this theorem, we show the following proposition.

Proposition 1. *There exists the unique matrix solution Z which is holomorphic in a neiborhood of the origin, so that $Z = I$ at the origin, satisfying the equations*

$$\left.\begin{array}{r} T_{x_1}Z(x_1,0,\dots,0) = Z(x_1,0,\dots,0)B_1(x_1,0,\dots,0) \\ T_{x_2}Z(x_1,x_2,0,\dots,0) = Z(x_1,x_2,0,\dots,0)B_2(x_1,x_2,0,\dots,0) \\ \cdots \\ T_{x_m}Z(x_1,\dots,x_m) = Z(x_1,\dots,x_m)B_{x_m}(x_1,\dots,x_m) \end{array}\right\} . \tag{11}$$

Proof. We fix $Z(\emptyset) = I$ as the constant term. We put further $Z_k = Z(x_1,\dots,x_k,0,\dots,0)$ such that $Z = Z_m$. Then the equations (11) are equivalent to the following

$$T_{x_k}Z_k = Z_kB_k(x_1,\dots,x_k,0,\dots,0)\,, \tag{12}$$

$$Z_k(x_1,\dots,x_{k-1},0) = Z_{k-1}(x_1,\dots,x_{k-1})\,. \tag{13}$$

We prove the Proposition by induction on $Z_j, j = 1,2,\dots,m$. We first consider Z_1. Assume that Z_1 has the expansion

$$Z_1 = I + \sum_{l=1}^{\infty} Z_1(\underbrace{1\dots1}_{l})x_1^l\,.$$

Then $Z_1(\underbrace{1\dots1}_{l})$, $(l \geq 1)$ satisfy the recurrence relations

$$q^lZ_1(\underbrace{1\dots1}_{l}) = \sum_{l',l''} Z_1(\underbrace{1\dots1}_{l'})B_1(\underbrace{1\dots1}_{l''})\,, \tag{14}$$

where l', l'' move over the set of non-negative integers such that $l = l' + l''$.

Hence $Z_1(\underbrace{1\ldots1}_{l})$ can be determined in a unique way starting from

$$Z_1(1) = -\frac{B_1(1)}{1-q}\,.$$

Suppose that $Z_j,\ j < k \leq m$ can be determined in a unique way from (11). We want to prove that Z_k can also be uniquely determined by (12), (13).

Equation (12) gives the recurrence relations for $Z_k(J),\ J = \{\underbrace{k\ldots k}_{l}, J'\}, (l \geq 1)$ such that $\{k, k+1, \ldots, m\} \cap J' = \emptyset$

$$q^l Z_k(\underbrace{k\ldots k}_{l}, J') = \sum_{l', l'', K', L'} Z_k(\underbrace{k\ldots k}_{l'}, K') B_k(\underbrace{k\ldots k}_{l''}, L')\,, \tag{15}$$

where l', l'' move over the non-negative integers such that $l' + l'' = l$ and K', L' move over the set of indices such that $J' = K' + L'$,$\{k, k+1, \ldots, m\} \cap K' = \emptyset$, $\{k, k+1, \ldots, m\} \cap L' = \emptyset$ respectively.

Hence $Z_k(\underbrace{k\ldots k}_{l}, J'),\ l \geq 1$ can be uniquely determined by the equation (12) starting from

$$Z_k(J') = Z_{k-1}(J'),\ Z_k(k, J') = -\frac{1}{1-q}\sum_{K'+L'=J'} Z_{k-1}(K')B_k(k, L')$$

for $J' \cap \{k, k+1, \ldots, m\} = \emptyset$ we mark that $B_k(J) = 0$ if $\{k\} \cap J = \emptyset$, see Lemma 2.1). It is obvious that $Z = Z_m$ thus constructed satisfies the equations (11) at least as formal power series.

Now we want to prove by induction that every $Z_k(x_1, \ldots, z_k)$ converges at the origin. From (8) we also have

$$B_k^{-1}(x) - I \equiv 0 \mod (x_k)\,. \tag{16}$$

Assume that $B_k^{-1}(x_1, \ldots, x_k, 0, \ldots, 0)$ has the expansion

$$B_k^{-1}(x_1, \ldots, x_k, 0, \ldots, 0) = \sum_{J, J \cap \{k+1, \ldots, m\} = \emptyset} \tilde{B}_k(J) x_J\,. \tag{17}$$

$B_k^{-1}(x)$ being holomorphic at the origin, we may assume that there exists a large positive number h such that every component of $\tilde{B}_k(J)$ is dominated by $h^{|J|}$, so that $\tilde{B}_k(J)$ as a matrix has the majorant

$$\tilde{B}_k(J) \prec\prec h^{|J|}\Omega^{|J|}\,,$$

where Ω denotes the matrix whose component is all equal to 1, i.e., $\Omega = \sum_{ij=1}^{s} E_{ij}$.

The equations (14) and (15) are equivalent to the following :

$$Z_1(\underbrace{1\dots 1}_{l}) = \sum_{l',l''} q^{l'} Z_1(\underbrace{1\dots 1}_{l'})\tilde{B}_1(\underbrace{1\dots 1}_{l''}) \tag{18}$$

$$Z_k(\underbrace{k\dots k}_{l}, J') = \sum_{l',l'',K',L'} q^{l'} Z_k(\underbrace{k\dots k}_{l'}, K')\tilde{B}_k(\underbrace{k\dots k}_{l''}, L') \tag{19}$$

respectively.

We now define the rational functions

$$b_k(x) = \prod_{J', J'\cap\{k,k+1,\dots,m\}=\emptyset} (1 - h^{1+|J'|}x_k x_{J'})\,.$$

Then the infinite product

$$f_k(x_1,\dots,x_k) = \prod_{J', J'\cap\{k,k+1,\dots,m\}=\emptyset} \frac{1}{(h^{1+|J'|}x_k x_{J'};q)_\infty}$$

satisfies the q-difference equations

$$T_{x_k} f_k(x_1,\dots,x_k) = f_k(x_1,\dots,x_k) b_k(x)\,.$$

In view of (16), we may assume that every component of the matrices $B_k^{-1}(x_1,\dots,x_k,0,\dots,0)$ has the following majorant series

$$B_k^{-1}(x_1,\dots,x_k,0,\dots,0) \prec\prec b_k^{-1}(\Omega x_1,\dots,\Omega x_k) \tag{20}$$

by taking a sufficiently large positive constant h.

Assume first $k=1$. Then every component of the matrix $B_1^{-1}(x_1,0,\dots,0)$ has the majorant series $\sum_{l=0}^{\infty} h^l x_1^l = \frac{1}{1-hx_1}$:

$$B_1^{-1}(x_1,0,\dots,0) \prec\prec \sum_{l=0}^{\infty} \Omega^l h^l x_1^l = \frac{1}{1-\Omega h x_1}$$

for a sufficiently large constant $h \gg 1$.

Hence by solving (18), one can see that every component of $Z_1(x_1)$ has the majorant $f_1(x_1)$:

$$Z_1(x_1) \prec\prec f_1(\Omega x_1) = \frac{1}{(\Omega h x_1;q)_\infty}\,.$$

By induction, by using (19), one can prove that every component of $Z_k(x_1,\dots,$ $x_k)$ has the majorant $\prod_{j=1}^{k} f_j(x_1,\dots,x_j)$:

$$Z_k(x_1,\dots,x_k) \prec\prec \prod_{j=1}^{k} f_j(\Omega x_1,\dots,\Omega x_j)\,.$$

Namely $Z_k(x_1,\dots,x_k)$ is a convergent series at the origin provided $|x_1|,\dots,$ $|x_m|$ are sufficiently small. By putting $k=m$ Proposition 1 is now true. □

Proof of Theorem 2. We want to prove that the function matrix Z in Proposition satisfies (10). Put

$$\tilde{Z} = T_{x_j} Z \cdot B_j^{-1}(x_1, \ldots, x_m)\,.$$

It is sufficient to show that $\tilde{Z}$ coincides with Z. $\tilde{Z}$ satisfies the equations (11). In fact for an arbitrary k, $1 \le k \le m$,

$$\begin{aligned}
& T_{x_k}\tilde{Z}(x_1, \ldots, x_k, 0, \ldots, 0) \\
= \;& T_{x_k}T_{x_j}Z_k \cdot \{T_{x_k}B_j\}^{-1}(x_1, \ldots, x_k, 0, \ldots, 0) \\
= \;& T_{x_j}\{T_{x_k}Z_k \cdot B_k^{-1}(x_1, \ldots, x_k, 0, \ldots, 0)\} \\
\cdot \;& \{T_{x_j}B_k(x_1, \ldots, x_k, 0, \ldots, 0) \cdot (T_{x_k}B_j)^{-1}(x_1, \ldots, x_k, 0 \ldots, 0)\} \\
= \;& T_{x_j}Z_k \cdot B_j^{-1}(x_1, \ldots, x_k, 0, \ldots, 0) \cdot B_k(x_1, \ldots, x_k, 0, \ldots, 0) \\
= \;& \tilde{Z}B_k(x_1, \ldots, x_k, 0, \ldots, 0)
\end{aligned}$$

because of the identities

$$\begin{aligned}
& B_k(x_1, \ldots, x_k, 0, \ldots, 0) \cdot T_{x_k}B_j(x_1, \ldots, x_k, 0, \ldots, 0) \\
&= B_j(x_1, \ldots, x_k, 0, \ldots, 0) \cdot T_{x_j}B_k(x_1, \ldots, x_k, 0, \ldots, 0), \\
& T_{x_k}Z_k = Z_kB_k(x_1, \ldots, x_k, 0, \ldots, 0)\,.
\end{aligned}$$

Moreover by definition $\tilde{Z} = I$ for $x = 0$. Hence by uniqueness it holds $\tilde{Z} = Z$, i.e.,

$$T_{x_j}Z = ZB_j(x)\,.$$

Theorem has been completely proved. □

Theorem 1 is an immediate consequence of Theorem 2:

Proof of Theorem 1. First note that each of A_k has the Laurent expansion

$$A_k = -(x_k + 1/x_k)I + A(\emptyset) + \sum_{r=1}^{\infty} \sum_{J,|J|=r} A(J)x_J\,.$$

Accordingly each of B_k has the series expansion

$$B_k = (1 + x_k^2)I - x_kA(\emptyset) - x_k \sum_{r=1}^{\infty} \sum_{J,|J|=r} A(J)x_J\,. \tag{21}$$

Hence in the equations (10) we have $B_k(k) = -A(\emptyset)$ and $B_k(j_1j_2) = 0$ except for the case $j_1 = j_2 = k$ where $B_k(kk) = I$. The matrix $A(\emptyset)$ is diagonalizable by a constant matrix P which is non-singular in the following form:

$$A(\emptyset) = P\Lambda P^{-1}\,,$$

where Λ denotes the diagonal matrix with the eigenvalues $\cos j\pi/(s+1)$, $j =$

$1, 2, \ldots, s$. The q-difference equation

$$T_{x_k}\mathbf{w} = \mathbf{w}(I - x_k A(\emptyset)) \tag{22}$$

has the fundamental solution matrix

$$W = \left(\sum_{j=1}^{s}\{\prod_{k=1}^{m}(\cos(j\pi/(s+1))x_k; q)_\infty\}E_{jj}\right) P^{-1}\,.$$

The solution matrix Z to (10) can then be expressed as

$$Z = WU^{-1}\,,$$

where U has the expansion (3) in a neighborhood of the origin. Hence

$$A_k = -B_k/x_k = -UW^{-1}T_{x_k}W \cdot \{T_{x_k}U\}^{-1} = U(I/x_k - A(\emptyset))\{T_{x_k}U\}^{-1}\,.$$

Theorem 1 is now true. □

Since the matrix $Y = \{\prod_{k=1}^{m}\theta(x_k; q)\}WU^{-1}$ defined in a neighbourhood of the origin satisfies the equation (2), it can be extendable on the whole space $(\mathbf{C}^*)^n$ as a meromorphic function matrix. Hence we can conclude:

Theorem 3. *The q-difference equation* (2) *has the fundamental solution Y meromorphic on $(\mathbf{C}^*)^n$, having the asymptotic behaviour Z in a neighbourhood of the origin:*

$$Y = \{\prod_{k=1}^{m}\theta(x_k; q)\}WU^{-1} \approx \{\prod_{k=1}^{m}\theta(x_k; q)\}W\,.$$

Corollary 1. *We define the product matrix*

$$G(x_1, \ldots, x_m) = (T_{x_{m-1}}\cdots T_{x_1}A_m)\cdots(T_{x_1}A_2)A_1\,,$$

then Y can be expressed by the infinite product

$$Y = \lim_{N\to\infty}\{\prod_{k=1}^{m}\theta(x_k q^N; q)\}G^{-1}(x_1 q^{N-1}, \ldots, x_m q^{N-1})\cdots G^{-1}(x_1, \ldots, x_m)\,.$$

Remark 2.2. Let τ_{jk} be the transposition between the elements x_j, x_k. Then the matrix $\tau_{jk}Y$ also satisfies the equations (2). Hence by uniqueness we have the relation

$$\tau_{jk}Y = Y\,,$$

i.e., Y is a symmetric function with respect to $x_1, \ldots, x_m$.

Finally one may pose the following question unknown to the author. This is a special connection problem (see [3], [6] for a general setting).

Question 2.3. The Jackson integrals $\langle\varphi_k, \xi\rangle$ may be connected with the fundamental solution Y :

$$(\langle\varphi_1,\xi\rangle,\ldots,\langle\varphi_s,\xi\rangle)=(c_1,\ldots,c_s)Y\,,$$

where c_k denote pseudo constants, i.e., q-periodic with respect to not only $x_1,\ldots,x_m$ but also ξ. Can c_k be evaluated explicitly?

Remark 2.4. Recently more precise local normal forms of q-difference equations with respect to one varibale has been obtained in [12], [13]. It seems interesting to have local normal forms which are holonomic q-difference equations extending them to many variables.

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