International Journal of Pure and Applied Mathematics
Volume 50 No. 2 2009, 151-156

SMOOTHING OF ROLL WAVE SHOCKS BY SURFACE TENSION

S.B.G. O'Brien[1 §], M. Robinson[2]
[1,2]Department of Mathematics
Mathematics Applications Consortium
for Science and Industry – MACSI
College of Science and Engineering
University of Limerick
Limerick, IRELAND

1. Introduction

The existence of roll wave solutions to the St. Venant equations has been investigated by several authors following the work of [2] who demonstrated the existence of a one parameter family of discontinuous (shock type) solutions. These waves are a manifestation of a supercritical bifurcation as suggested by a straightforward linear stability analysis. The Dressler model was extended by [4] to include, in an ad-hoc way, an eddy viscosity energy dissipation term which allowed *continuous* periodic solutions. Several authors (e.g. [1]) have alluded to the possibility of including surface tension effects in the model, with the implication that such terms will make a contribution to the smoothing of the roll-wave solutions. Here, we formally include the effects of surface tension and demonstrate that, while such terms in the presence of eddy viscosity smoothing terms undoubtedly influence the details of the shock structure, they cannot on their own give rise to continuous roll-wave solutions.

Received: August 14, 2008
§Correspondence author

2. Model Description

We model the flow of water down an inclined channel. The water surface $z = \eta(x,t)$ is assumed to be horizontal in any cross-section. The water depth $h(x,y,t)$ varies in both the downstream and horizontal directions. Thus, the cross-sectional area of the channel is $A(x,t) = \int h \; dy$ and the channel bed is located at $z = \eta - h$. A full derivation of the one-dimensional St. Venant equations is given by Stoker (1957) and we write them here in the form

$$A_t + (Au)_x = 0, \tag{1}$$

$$(Au)_t + (Au^2)_x = gA\sin\alpha - \frac{Afu^2}{R} - \frac{1}{\rho}\bar{p}_x, \tag{2}$$

where $u(x,t)$ is the averaged velocity, α is the channel inclination, $R = \frac{A}{l}$ is the hydraulic radius, l is the wetted perimeter of a cross-section and f is a friction factor depending on the roughness of the channel walls and bed. The cross-sectional mean pressure $\bar{p}$ satisfies $\bar{p} = \int\int p \; dzdy$. To determine $\bar{p}$ we consider the forces at the air-water interface $z = \eta$. The upward force on the surface water is p and the downward force exerted by the air is p_a. Thus, the boundary condition at the interface is $p - p_a = \gamma\kappa$, where γ is surface tension and κ is the curvature of the free surface which satisfies $\kappa = -\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}$. A uniform level free surface implies $\eta_x \ll 1$ and taking $p_a = 0$ the boundary condition at the interface will be $p = -\gamma\eta_{xx}$. Pressure is approximately hydrostatic satisfying $p = -\rho g z + c(x,t)$ and an application of the interfacial boundary condition yields $p = \rho g(\eta - z) - \gamma\eta_{xx}$. Furthermore, for a relatively smooth channel bed h_x would be independent of y and $\bar{p}_x = \rho g A h_x - \gamma(A\eta_{xx})_x$ can be readily calculated and equation (2) can be written as

$$u_t + uu_x = g\sin\alpha - \frac{fu^2}{R} - gh_x + \frac{\gamma}{\rho A}(A\eta_{xx})_x. \tag{3}$$

2.1. Non-Dimensionalisation and Simplification

We start by assuming an approximately uniform channel bed such that $\eta \approx h$ and define dimensionless variables as follows

$$x = \frac{d}{\sin\alpha}x^*, \quad A = a_0 A^*, \quad u = u_0 u^*, \quad t = \frac{d}{u_0\sin\alpha}t^*, \quad h = dh^*, \quad R = R_0 R^*,$$

where d is the average channel depth. If we assume a wide rectangular channel, with constant width w, then $w \gg h$. The wetted perimeter l is then $l = w + 2h \approx w$ and the cross-sectional area satisfies $A = wh \approx lh$. The dimensional hydraulic radius is then given by $R \approx h \approx \frac{A}{w}$ and it follows that suitable scale values are $a_0 = wd$ and $R_0 = d$. The velocity scale $u_0 = \sqrt{\frac{gd \sin\alpha}{f}}$ is chosen to balance gravitational and frictional effects. A standard non-dimensionalisation of equations (1) and (3) yields

$$h^*_{t^*} + (h^* u^*)_{x^*} = 0, \tag{4}$$

$$Fr^2 \left(u^*_{t^*} + u^* u^*_{x^*}\right) = 1 - \frac{{u^*}^2}{h^*} - h^*_{x^*} + \frac{\beta}{h^*}(h^* h^*_{x^* x^*})_{x^*}, \tag{5}$$

where $\beta = \frac{\gamma \sin^2\alpha}{g\rho d^2}$ and $Fr = \sqrt{\frac{\sin\alpha}{f}}$ is the Froude number.

3. Results

3.1. Solutions with Eddy Viscosity Effects

Neglecting surface tension effects (i.e. $\beta = 0$), the St. Venant system is hyperbolic resulting in shock formation in the developing waves. The existence of continuous solutions has been shown by [4] with the inclusion of a term of the form $\nu_0 u_{xx}$ to account for energy dissipation by shearing normal to the flow, where ν_0 is an eddy viscosity. In this section we show how the addition of such a term directly affects the structure of the underlying shock. Dropping asterisks, the dimensionless equations we consider are

$$h_t + (hu)_x = 0,$$

$$Fr^2 \left(u_t + u u_x\right) = 1 - \frac{u^2}{h} - h_x + \varepsilon u_{xx},$$

where, $\varepsilon = \frac{\nu_0 \sin^2\alpha}{f d u_0}$. The viscosity term will have its greatest influence close to the shock where the gradients are large. Consequently, if the shock location is denoted by $x = x_s(t)$, we let $x = x_s(t) + \delta\zeta$, where $\delta \ll 1$ is the shock width. Choosing $\delta = \varepsilon$ we find that $h(\zeta)$ satisfies

$$2C_1 h_\zeta = -\left(2h^3 + Fr^2 C_1^2 - 2C_2 h^2\right), \tag{6}$$

where C_1 and C_2 are constants. The solution of (6) will only be valid in a small neighbourhood of the shock and Dressler's solution should be approximately valid far from the shock. Hence, a valid shock structure should match the *outer* solution (when $\varepsilon = 0$) with the *inner* shock structure. This is achieved by prescribing the matching boundary conditions,

$$h \to h_L \quad \text{as} \quad \zeta \to -\infty, \qquad h \to h_0 \quad \text{as} \quad \zeta \to +\infty, \tag{7}$$

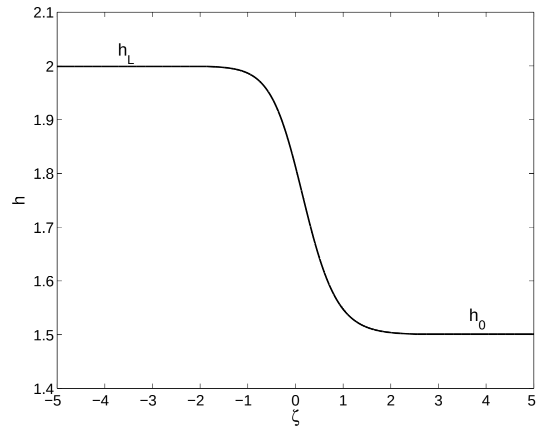

Figure 1: $h(\zeta)$ with $Fr = 5$, $h_L = 2$, $h_0 = 1.5$ and $h(0) = 1.8$.

where h_L and h_0 are the values of h before and after the shock respectively. Boundary conditions (7) allow determination of the constants C_1 and C_2, however, these conditions require derivatives of $h(\zeta)$ to vanish in the limits as $\zeta \to \pm\infty$ and in order that (6) have a solution satisfying the boundary conditions (7) we require $h = h_0$ and $h = h_L$ be roots of the cubic and we rewrite (6) in the form

$$C_1 h_\zeta = -(h - h_0)(h - h_L)(h - \psi),$$

where $\psi = -\frac{h_0 h_L}{h_0 + h_L}$. A direct integration yields the implicit solution

$$\frac{\ln|h - h_0|}{(h_0 - h_L)(h_0 - \psi)} + \frac{\ln|h - h_L|}{(h_0 - h_L)(\psi - h_L)} + \frac{\ln|h - \psi|}{(\psi - h_L)(\psi - h_0)} = -\frac{\zeta}{C_1} + C_3,$$

where C_3 is an integration constant, which can be determined for a given initial condition. We can plot $h(\zeta)$ for prescribed values of h_0, h_L and Fr. Figure 1 shows such a solution and the smooth transition from h_L to h_0 in the shock region $\zeta \approx 0$ is clearly visible.

3.2. Solutions with Surface Tension Effects

We now investigate if surface tension effects alone can yield continuous solutions. The hyperbolic case was examined in [2], and discontinuous periodic solutions exist when the Froude number satisfies $Fr > 2$. A straightforward linear stability analysis finds the criterion for instability unchanged when $\beta \neq 0$ and disturbances with wavenumbers satisfying $k^2 < \frac{Fr^2 - 4}{4\beta}$ become unstable. As

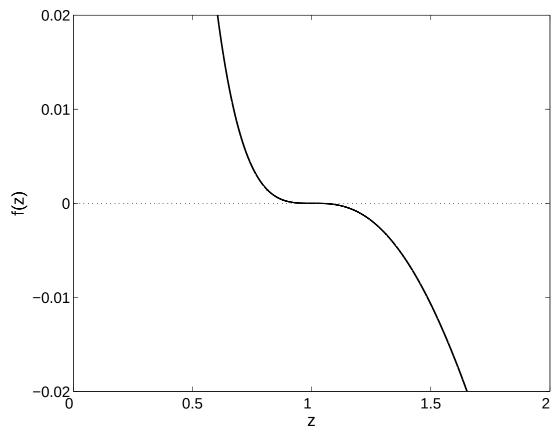

Figure 2: Plot of $f(z)$ as defined by (11)

before, we consider the effects of this term on the shock and, choosing $\delta = \sqrt{\beta}$, we obtain a second-order differential equation for $h(\zeta)$,

$$2h^2 h_{\zeta\zeta} = h^3 + 2Fr^2 C_4^2 - 2C_5 h, \tag{8}$$

where C_4 and C_5 are constants. As before, the matching boundary conditions

$$h \to h_L \quad \text{as} \quad \zeta \to -\infty, \qquad h \to h_0 \quad \text{as} \quad \zeta \to +\infty, \tag{9}$$

allow determination of the constants. Now, assuming (8) has a solution satisfying the conditions (9) we can write it in the form

$$2h^2 h_{\zeta\zeta} = (h - h_0)(h - h_L)(h + h_0 + h_L),$$

which can be integrated yielding

$$h_\zeta^2 = \frac{h^2}{2} - (h_0^2 + h_0 h_L + h_L^2)\ln(h) - \frac{h_0 h_L (h_0 + h_L)}{h} + C_6, \tag{10}$$

and C_6 can be determined in terms of h_0 and h_L by the requirement that h_ζ vanish as $h \to h_L$. Then for correct shock behaviour, the substitution $h = h_0$ in (10) should yield $h_\zeta = 0$ to ensure that h_ζ also vanish in the limit $h \to h_0$. After some algebra, this requires satisfaction of the equation

$$f(z) = \frac{\frac{3}{2}(z^2 - 1)}{z^2 + z + 1} - \ln(z) = 0, \quad \text{where} \quad z = \frac{h_0}{h_L}. \tag{11}$$

Figure 2 clearly shows that $f(z) \neq 0$ for all $z \neq 1$ and the equality (11) can only be satisfied when $z = 1$ (i.e. $h_0 = h_L$) corresponding to no shock. We thus conclude that equation (8) possesses no solutions that satisfy the boundary conditions (9) and the addition of surface tension effects alone to the St. Venant equation system does not yield continuous roll-wave solutions.

From a mathematical point of view, the inability of surface tension effects to smooth the roll-wave solutions lies in the fact that equation (8) represents a conservative autonomous Hamiltonian system. Letting $v = h_\zeta$ there exists a function $H(h, v)$ such that

$$\frac{\partial H}{\partial h} = -v_\zeta \quad \text{and} \quad \frac{\partial H}{\partial v} = h_\zeta, \quad \text{where} \quad H = \frac{v^2}{2} - \frac{h^2}{4} + \frac{Fr^2 C_4^2}{h} + C_5 \ln h.$$

4. Conclusion

The inability of surface tension to produce a shock stucture is surprising. In thin film (lubrication) flows, it is well known (see [3], [5]) that shocks typically can be smoothed via second derivative terms (originating from the hydrostatic pressure gradient) and/or surface tension higher derivative terms.

References

[1] N.J. Balmforth, J.J. Liu, Roll-waves in mud, *J. Fluid Mech.*, **519** (2004), 33-54.

[2] R.F. Dressler, Mathematical solution of the problem of roll-waves in inclined open channels, *Comm. Pure Appl. Maths.*, **2** (1949), 149-194.

[3] J.A. Moriarty, L.W. Schwartz, E.O. Tuck, Unsteady spreading of thin liquid films with small surface tension, *Phys. Fluids, A*, **3** (1991), 733-743.

[4] D.J. Needham, J.H. Merkin, On roll-waves down an open inclined channel, *Proc. R. Soc. Lond.*, **394** (1984), 259-278.

[5] S.B.G. O'Brien, On Marangoni drying: Non-linear kinematics waves in a thin film, *J. Fluid Mech.*, **254** (1993), 49-70.