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# NONCOMMUTATIVE SOLITONS AND KINKS IN THE AFFINE TODA MODEL COUPLED TO MATTER

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**Abstract:** Some properties of the non-commutative (NC) versions of the generalized sine-Gordon model (NCGSG) and its dual massive Thirring theory are studied. Our method relies on the NC extension of integrable models and the master lagrangian approach to deal with dual theories. The master lagrangian turns out to be the NC version of the so-called affine Toda model coupled to matter related to the group $GL(n)$, in which the Toda field $g \subset GL(n)$ $(n = 2, 3)$. Moreover, as a reduction of $GL(3)$ NCGSG one gets a NC version of the remarkable double sine-Gordon model.

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## 1. Introduction

Some non-commutative versions of the sine-Gordon model (NCSG) have been

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proposed in the literature [9, 6, 8]. The relevant equations of motion have the general property of reproducing the ordinary sine-Gordon equation when the non-commutativity parameter is removed. It has been shown that these models arise as reduced models starting from a master Lagrangian, the so-called $sl_2$ affine Toda model coupled to matter [6]. In the present paper we summarize those results, as well as certain new NC versions of the (generalized) sine-Gordon and massive Thirring dual models [2].

In the following we consider an algebra of continuous functions with the Moyal product or star product

$$f \star g(x) \;\; = \;\; \exp\Big(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{(x')}\partial_\nu^{(x'')}\Big) f(x')g(x'')|_{x'=x''=x}. \tag{1}$$

So, one has the coordinate noncommutativity, i.e., $\Big[\hat{x}^\mu, \hat{x}^\nu\Big]_\star = i\theta^{\mu\nu}$.

## 2. Sine-Gordon/Massive Thirring Models

The sine-Gordon action is defined by

$$S = \frac{1}{2}\int \; dt\, dx \big[\partial_\mu\varphi\partial^\mu\varphi + \frac{2\alpha_0}{\beta^2}(\cos\beta\varphi - 1)\big] \tag{2}$$

for a scalar field $\varphi(t,x)$ on $\mathbb{R}^{1,1}$. This model has many remarkable features, such as a Lax-pair representation, infinitely many conserved local charges, a factorizable S-matrix, as well as soliton and breather solutions. The simplest soliton configuration is

$$\varphi_{\text{kink}}(t,x) = 4\text{arctan}\exp[-\frac{\alpha_0}{2}\xi] \tag{3}$$

with $\xi = \frac{x-vt}{\sqrt{1-v^2}}$ .

In the usual space-time the quantum equivalence between the massive Thirring and sine-Gordon models is well known. The Thirring model

$$S_{NCMT} = \int dtdx\big[i\bar{\psi}\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi - \frac{\lambda}{2}j^\mu j_\mu\big]\,, \qquad j^\mu = \bar{\psi}\gamma^\mu\psi \tag{4}$$

is equivalent to the sine-Gordon model provided that $\frac{4\pi}{\beta^2} = 1 + \frac{\lambda}{\pi}$. The same equivalence holds in NC space-time even though their actions look very different. However, the coupling relationship maintains its form up to some rescalings [3]. In the following we provide the classical NC counterparts of the SG and MT models, as well as some multi-field generalizations.

## 3. The Master Lagrangian

The NC affine Toda model coupled to matter (spinor) (NCATM) is defined by

$$S_{NCATM} \equiv S[g, W^{\pm}, F^{\pm}] = I_{WZW}[g] + \int d^2x \{ \frac{1}{2} Tr[\partial_- W^- \star [E_2\, , \, W^-]] - \frac{1}{2} Tr[[E_{-2}\, , \, W^+] \star \partial_+ W^+] + Tr[F^- \star \partial_+ W^+] + Tr[\partial_- W^- \star F^+] + Tr[F^- \star g \star F^+ \star g^{-1}] \}, \quad (5)$$

where $F \star G = F \exp\Big(\frac{\theta}{2}(\overleftarrow{\partial_+}\overrightarrow{\partial_-} - \overleftarrow{\partial_-}\overrightarrow{\partial_+})\Big) G$, $E_{\pm 2}$ is a constant matrix. Consider $x_{\pm} = t \pm x$, then , $\partial_{\pm} = \frac{1}{2}(\partial_t \pm \partial_x)$. $I_{WZW}[g]$ is the *NC Wess-Zumino-Witten* model

$$I_{WZW}[g] = \int d^2x \partial_+ g \star \partial_- g^{-1} + \int_0^1 dy \hat{g}^{-1} \star \partial_y \hat{g} \star \Big[\hat{g}^{-1} \star \partial_+ \hat{g}, \hat{g}^{-1} \star \partial_- \hat{g}\Big]_{\star},$$

where $\hat{g}(y)$ is such that $\hat{g}(0) = 1$, $\hat{g}(1) = g$ $([y, x_+] = [y, x_-] = 0)$.

The matrix fields are defined by

$$E_{\pm 2} = \frac{m_{\psi}}{4} H^{\pm 1}, \;\; g[SG \text{ field(s)}] \in U(1) \times U(1) \;\text{ or }\; U(1)_{\mathbb{C}}$$

$$F^+ = \sqrt{i m_{\psi}} \Big(\psi_R E_+^0 + \widetilde{\psi}_R E_-^1\Big), \;\; F^- = \sqrt{i m_{\psi}} \Big(\psi_L E_+^{-1} - \widetilde{\psi}_L E_-^0\Big). \quad (6)$$

The affine Lie algebra $\hat{sl}(2)$ commutation relations are

$$[H^m\, , \, H^n] = 2\, m\, C\, \delta_{m+n,0}, \quad (7)$$

$$[H^m\, , \, E_{\pm}^n] = \pm 2\, E_{\pm}^{m+n}, \quad (8)$$

$$[E_+^m\, , \, E_-^n] = H^{m+n} + m\, C\, \delta_{m+n,0}. \quad (9)$$

The Dirac fields in components become

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}; \;\; \widetilde{\psi} = \begin{pmatrix} \widetilde{\psi}_R \\ \widetilde{\psi}_L \end{pmatrix} \quad (10)$$

The equations of motion for the NCATM model in matrix form are

$$\partial_+(\partial_- g \star g^{-1}) = \Big[F^-\, , \, g \star F^+ \star g^{-1}\Big]$$

$$\partial_+ F^- = -[E_{-2}, g \star F^+ \star g^{-1}] \quad , \quad \partial_- F^+ = -[E_2, g^{-1} \star F^- \star g].$$

A ‘local’ symmetry of the model [5, 2] allows one to view the field $g$ as a ‘gauge field’ and $F^{\pm}$ as matter fields. Due to that symmetry the NCATM model can be reduced, through some methods, to the $\text{NCSG}_{1,2}$ models.

### 3.1. NC Versions of the Sine-Gordon Model $NCSG_{1,2}$

The general effective action of the reduced bosonic model becomes

$$S[g] = I_{WZW}[g] + \int d^2x \{Tr[\Lambda^- \star g \star \Lambda^+ \star g^{-1}]\}. \qquad (11)$$

**First Version.** Consider $g \in U(1)\mathrm{x}U(1)$ in the representation

$$g = \begin{pmatrix} e_\star^{i\varphi_+} & 0 \\ 0 & e_\star^{-i\varphi_-} \end{pmatrix}, g_+ \equiv \begin{pmatrix} e_\star^{i\varphi_+} & 0 \\ 0 & 1 \end{pmatrix}, \;\; g_- \equiv \begin{pmatrix} 1 & 0 \\ 0 & e_\star^{-i\varphi_-} \end{pmatrix}$$

with $\varphi_\pm$ being real fields.

For the $\Lambda$'s taken as

$$\Lambda^+ = M(E_+^0 + E_-^1), \;\; \Lambda^- = M(E_-^0 + E_+^{-1}),$$

the action (11) can be written as

$$S_{NCSG_1}[g_+, g_-] = I_{WZW}[g_+] + I_{WZW}[g_-] + M^2 \int d^2x \Big(e_*^{i\varphi_+ + i\varphi_-} + e_*^{-i\varphi_+ - i\varphi_-} - 2\Big). \qquad (12)$$

The equations of motion become

$$\partial_+\Big(\partial_- e_\star^{i\pm\varphi_\pm} \star e_\star^{\mp i\varphi_\pm}\Big) = \mp M^2 \Big(e_\star^{i\varphi_-} \star e_\star^{i\varphi_+} - e_\star^{-i\varphi_+} \star e_\star^{-i\varphi_-}\Big).$$

This is the Lechtenfeld et al proposal for $\mathrm{NCSG}_1$ model. In the $\theta \to 0$ limit one has a free field equation, $\partial_-\partial_+\Big(\varphi\Big) = 0$; $(\varphi \equiv \varphi_+ - \varphi_-)$ and the usual SG equation $\partial^2 \varphi_{SG} = -4M^2 \sin(\varphi_{SG})$ $(\varphi_{SG} \equiv \varphi_+ + \varphi_-)$.

**Second Version.** Consider $g \in U(1)_{\mathbb{C}}$

$$g = e_\star^{i\varphi H^0} \equiv \begin{pmatrix} e_\star^{i\varphi} & 0 \\ 0 & e_\star^{-i\varphi} \end{pmatrix} \quad \text{and} \quad \bar{g} = g^\star, \qquad (13)$$

where the field $\varphi$ is a general complex field. The equations of motion are

$$\partial_+\Big(\partial_- e^{\mp \frac{i}{2}\varphi} \star e^{\pm \frac{i}{2}\varphi}\Big) = \pm \frac{\gamma}{4}\Big(e^{i\varphi} - e^{-i\varphi}\Big), \; \gamma = \text{const.} \qquad (14)$$

and similar equations for $\bar{g}$ considered as an independent field. This is the Grisaru-Penati proposal for the $\mathrm{NCSG}_2$.

## 4. NC Versions of the Massive Thirring Model

The dual to the $\mathrm{NCSG}_1$ model turns out to be the NC bi-fundamental $U(1) \times U(1)$ massive Thirring model [6, 3]. The (Euclidean) Lagrangian is

$$\mathcal{L}_{NCMT_1} = i\bar{\psi}\gamma^{\mu}\partial_{\mu} \star \psi + m\bar{\psi} \star \psi - \frac{\lambda_b}{4} j^{(A)\,\mu} \star j^{(A)}_{\mu} - \frac{\lambda_b}{4} j^{(B)\,\mu} \star j^{(B)}_{\mu}, \qquad (15)$$

defined for Dirac fields and with the currents given by

$$j^{(A)\,\mu} \equiv \bar{\psi} \star \gamma^{\mu}\psi; \;\; j^{(B)\,\mu} \equiv \psi_{\beta}(\gamma^{\mu})_{\alpha\beta} \star \bar{\psi}_{\alpha}. \qquad (16)$$

Here, $\lambda_b$ is the coupling constant and the group index contractions are being assumed. The currents correspond to the $U(1) \times U(1)$ symmetry in NC space. Two copies of the above $\mathrm{NCMT}_1$ model defines the relevant $\mathrm{NCMT}_2$ corresponding to the $\mathrm{NCSG}_2$.

## 5. The Generalized NC Sine-Gordon $\mathrm{NCGSG}_1$

Let us consider $g \subset GL(n)$, $n > 2$. We must take the bosonic action (11) for a relevant set $\{\Lambda^{\pm}_m, g\}$[5]. Consider the parametrization $g \in [U(1)]^3 \subset GL(3, \mathbb{C})$

$$g = \begin{pmatrix} e_{\star}^{i\phi_1} & 0 & 0 \\ 0 & e_{\star}^{i\phi_2} & 0 \\ 0 & 0 & e_{\star}^{i\phi_3} \end{pmatrix} \equiv g_1 * g_2 * g_3, \qquad (17)$$

with $\phi_i$ being real fields ($i = 1, 2, 3$). The equations of motion are

$$\partial_{+}\Big(\partial_{-} e_{\star}^{i\phi_i} \star e_{\star}^{-i\phi_i}\Big) = G_i, \quad i = 1, 2, 3\,,$$

$$G_1 \equiv \frac{M_1}{8}[e_{\star}^{i\phi_2} \star e_{\star}^{-i\phi_1} - e_{\star}^{i\phi_1} \star e_{\star}^{-i\phi_2}] + \frac{M_3}{8}[e_{\star}^{i\phi_3} \star e_{\star}^{-i\phi_1} - e_{\star}^{i\phi_1} \star e_{\star}^{-i\phi_3}]\,,$$

$$G_2 \equiv \frac{M_2}{8}[e_{\star}^{i\phi_3} \star e_{\star}^{-i\phi_2} - e_{\star}^{i\phi_2} \star e_{\star}^{-i\phi_3}] + \frac{M_1}{8}[e_{\star}^{i\phi_1} \star e_{\star}^{-i\phi_2} - e_{\star}^{i\phi_2} \star e_{\star}^{-i\phi_1}]\,,$$

$$G_3 \equiv \frac{M_3}{8}[e_{\star}^{i\phi_1} \star e_{\star}^{-i\phi_3} - e_{\star}^{i\phi_3} \star e_{\star}^{-i\phi_1}] + \frac{M_2}{8}[e_{\star}^{i\phi_2} \star e_{\star}^{-i\phi_3} - e_{\star}^{i\phi_3} \star e_{\star}^{-i\phi_2}].$$

This system defines the $\mathrm{NCGSG}_1$. In the $\theta \to 0$ limit one gets a free scalar

$$\partial^2\,\Phi \;\;=\;\; 0, \quad \Phi \equiv \phi_1 + \phi_2 + \phi_3.$$

For the particular solution $\Phi \equiv 0$, one can write

$$\begin{aligned} \partial^2\,\phi_1 &= M_1 \sin(\phi_2 + \phi_1) + M_3 \sin(2\phi_1 - \phi_2); \\ \partial^2\,\phi_2 &= M_2 \sin(2\phi_2 - \phi_1) + M_1 \sin(\phi_1 + \phi_2). \end{aligned} \qquad (18)$$

This is the usual generalized sine-Gordon model [4]. Remarkably, a version of the NC double sine-Gordon model ($NCDSG_1$) emerges for the reduction $\phi_1 = -\phi_3 = \phi,\ \phi_2 = 0$. So, one gets the equations

$$\partial_+\Big(\partial_- e_\star^{i\phi} \star e_\star^{-i\phi} + \partial_- e_\star^{-i\phi} \star e_\star^{i\phi}\Big) \quad = \quad 0, \tag{19}$$

$$\partial_+\Big(\partial_- e_\star^{-i\phi} \star e_\star^{i\phi}\Big) \quad = \quad 2iM_1 \sin_\star\phi + 2iM_3 \sin_\star 2\phi\,. \tag{20}$$

In the limit $\theta \rightarrow 0$ equation (20) reduces to $\partial_+\partial_-\phi = -2M_1 \sin\phi - 2M_3 \sin 2\phi$.

A second version $NCGSG_2$ exists for an another form of $g$, see [5].

## 6. Generalized NC Thirring model $NCGMT_1$

The dual to the $NCGSG_1$ model defined for three Dirac fields becomes [5]

$$\begin{aligned} \mathcal{L}_{NCGMT_1} \quad = \quad & i\sum_{j=1}^{3}\Big(\bar{\psi}_j\gamma^\mu\partial_\mu \star \psi_j + m_j\bar{\psi}_j \star \psi_j\Big) \\ & -\sum_{j}^{3} g_{jj}\Big(j_j^{(A)\,\mu} \star j_{\mu\,j}^{(A)} + j_j^{(B)\,\mu} \star j_{\mu\,j}^{(B)}\Big) \\ & +g_{12}\Big[j_1^{(A)\,\mu} \star j_{\mu\,2}^{(B)}\Big] - g_{23}\Big[j_2^{(A)\,\mu} \star j_{\mu\,3}^{(A)}\Big] - g_{13}\Big[j_1^{(B)\,\mu} \star j_{\mu\,3}^{(B)}\Big]\,, \end{aligned}$$

$$j_j^{(A)\,\mu} \equiv \bar{\psi}_j \star \gamma^\mu\psi_j;\;\; j_j^{(B)\,\mu} \equiv \psi_{j\,\beta}(\gamma^\mu)_{\alpha\beta} \star \bar{\psi}_{j\,\alpha}.$$

Here, $g_{jk}$ are the coupling constants and the group index contractions are being assumed.

A second version $NCGMT_2$, corresponding to the $NCGSG_2$ mentioned above, can be obtained by doubling the number of Dirac fields.

## 7. Discussion

The solitons and kinks of the NC sine-Gordon/massive Thirring models have been constructed in [4]. It is known that if $f(t,x)$ and $g(t,x)$ depend only on $(x - vt)$, then $f \star g \Rightarrow fg$. Therefore, all the $\star$ products for this type of functions (e.g. soliton and kink type solutions) become the same as the ordinary ones. So, $1-$soliton solutions of the usual GSG model are also solutions of the NC models. The NC extension of the SG/MT duality provides a rich structure already at the classical level. The NC actions differ from their commutative counterparts. By suitably reducing the $NCGSG_{1,2}$ models one can get some versions of the

NC double sine-Gordon model ($NCDSG_{1,2}$). Moreover, the $1-$kink solution of the DSG model solves its NC counterparts $NCDSG_{1,2}$. Several future research directions arise: multi-solitons, soliton scattering, quantum versions, S-matrix, and hopefully some physical applications.

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