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A CONSTRUCTION OF AN ORTHOGONAL BASIS IN SOME SOBOLEV SPACES

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Abstract: It is used the method of orthogonal sequences of Bergmann [2], [4] and Vekua [5], to find an orthogonal basis in the Sobolev space $H_0^1(D)$, where D is a quarter of circle. The elements of the basis are the solutions of some eigenvalue boundary problem.

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1. Introduction

In practise arise real difficulties in the problem of finding a base in Hilbert spaces. We give here a method of elimination of these difficulties using Bergmann's method of double orthogonal sequences [2], [3], [4].

Let $(H, (\cdot,\cdot))$, $(V, <\cdot,\cdot>)$ be real, separable Hilbert spaces and denote by $\|\cdot\|$, $|\cdot|$ the corresponding norms, respectively. In what follows, we use the next result due to Bergmann [2].

Theorem 1. *Assume that $H \subset V$ and the imbedding $H \hookrightarrow V$ is compact,*

$$|x| \le c\,\|x\|$$

for every $x \in H$ and for some positive constant c. Then there exist an increasing,

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unbounded sequence $(\lambda_n)_{n\geq 1}$ *of positive reals and a sequence* $(e_n)_{n\geq 1} \subset H$ *which is orthogonal with respect to both inner products, i.e.*

$$(e_m, e_n) = \lambda_n \delta_{mn} \quad , \quad < e_m, e_n >= \delta_{mn} \, , \tag{1.1}$$

for all positive integers m, n. *Moreover,* $(e_n)_{n\geq 1}$ *is complete in* H.

Theorem 1 is in fact a method to find an orthogonal basis in H where the elements of the basis are the solutions of some optimization problems. Remark that from (1.1), we can derive the equalities

$$(e_m, e_n) = \lambda_n < e_m, e_n >,$$

for all $m, n \geq 1$. Because of completness of the system $(e_n)_{n\geq 1}$, it follows that

$$(e_n, v) = \lambda_n < e_n, v >, \tag{1.2}$$

for every $n \geq 1$, $v \in H$. In consequence, the elements of the orthogonal basis $(e_n)_{n\geq 1}$ can be considered as the solutions of the eigenvalue problem (1.2). In fact, this is an useful method to find a basis in a real separable Hilbert space, as we can see below.

The set of all functions $u \in L^2(D)$ with $u = 0$, on ∂D, having generalized derivative is denoted by $H_0^1(D)$. The space $H_0^1(D)$ also called Sobolev space is a Hilbert space relative to the scalar product

$$(u, v) = \int_D uv + \int_D \nabla u \nabla v, \quad u, v \in H_0^1(D)$$

with the corresponding norm

$$\|u\| = \left(\int_D u^2 + \int_D |\nabla u|^2 \right)^{1/2}, \quad u \in H_0^1(D).$$

Consider also the Hilbert space $L^2(D)$ endowed with the usual scalar product

$$< u, v >= \int_D uv, \quad u, v \in L^2(D)$$

and the usual norm

$$|u| = \left(\int_D u^2 \right)^{1/2}, \quad u \in L^2(D).$$

The imbedding

$$H_0^1(D) \hookrightarrow L^2(D)$$

is compact because

$$|u| \leq \|u\| \, , \quad \text{for every} \ \ u \in H_0^1(D).$$

In order to give a method to find an orthogonal basis in $H_0^1(D)$, we will use

Theorem 1. The eigenvalue problem (1.2) can be written as

$$\int_D e_n v + \int_D \nabla e_n \nabla v = \lambda_n \int_D e_n v\,, \quad \text{for every} \;\; v \in H_0^1(D),\ n \geq 1. \qquad (1.3)$$

But $v = 0$, on ∂D, so

$$\int_D \nabla e_n \nabla v = -\int_D v \Delta e_n,$$

if e_n is twice derivable. Hence (1.3) is equivalent with

$$\int_D e_n v - \int_D v \Delta e_n = \lambda_n \int_D e_n v,$$

or

$$\int_D (\Delta e_n + (\lambda_n - 1) e_n) v = 0\,, \quad \text{for every} \;\; v \in H_0^1(D).$$

We deduce that $(e_n)_{n\geq 1}$ are the eigenfunctions of the next boundary problem

$$\begin{cases} \Delta u(x) = (\lambda - 1) u(x)\,, & \text{in} \;\; D\,, \\ \dfrac{\partial u}{\partial n}(x) = 0\,, & \text{on} \;\; \partial D\,. \end{cases} \qquad (1.4)$$

2. The Result

Let $D \subset \mathbb{R}^2$ be a domain having the boundary with a corner of angle π/k, where $k \geq 1/2$. A method for finding the eigenfunctions for the Laplacian is Bergman-Vekua method (e.g. [2]) which gives

$$u(r,t) = \sum_{j=1}^{N} c_j J_{jk}(\sqrt{\lambda r}) \sin jkt, \qquad (2.1)$$

where (r,t) are polar coordinates, J_β is Bessel function of order β and c_j and λ will be determined.

We will consider the domain D as the semiquarter of the circle ($k = 4$) with radius $r = 1$. For sake of simplicity we will impose the boundary conditions only for the points $P_1(1, \pi/6)$ and $P_2(1, \pi/12)$.

In this case, $N = 2$ and the solution is given by the formula

$$u(r,t) = c_1 J_4(\sqrt{\lambda r}) \sin 4t + c_2 J_8(\sqrt{\lambda r}) \sin 8t. \qquad (2.2)$$

The boundary condition $\dfrac{\partial u}{\partial n}(x) = 0$ on ∂D is equivalent with

$$\frac{\partial u}{\partial x} \cdot \cos t + \frac{\partial u}{\partial y} \cdot \sin t = 0 \;\; \text{on} \;\; \partial D \qquad (2.3)$$

and we use the variable changes

$$x = r\cos t, \qquad y = r\sin t\,,$$

$0 \le r \le 1$, $0 \le t \le \pi/4$. Thus

$$r = \sqrt{x^2+y^2}, \qquad t = \arcsin\frac{y}{\sqrt{x^2+y^2}}\,,$$

then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} = \frac{\partial u}{\partial r}\cdot\cos t + \frac{\partial u}{\partial t}\cdot r\sin t\,, \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} = \frac{\partial u}{\partial r}\cdot\sin t + \frac{\partial u}{\partial t}\cdot\frac{\cos t}{r}\,.\end{aligned} \tag{2.4}$$

It follows from (2.2):

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{c_1\lambda}{2\sqrt{\lambda r}}\cdot J_4'(\sqrt{\lambda r})\sin 4t + \frac{c_2\lambda}{2\sqrt{\lambda r}}\cdot J_8'(\sqrt{\lambda r})\sin 8t\,, \\ \frac{\partial u}{\partial t} &= 4c_1\cdot J_4(\sqrt{\lambda r})\cos 4t + 8c_2\cdot J_8(\sqrt{\lambda r})\cos 8t\,.\end{aligned} \tag{2.5}$$

With (2.4), the boundary condition (2.3) becomes

$$\frac{\partial u}{\partial r} + \frac{1}{2}\left(r+\frac{1}{r}\right)\cdot\frac{\partial u}{\partial t}\cdot\sin 2t = 0 \quad \text{on} \quad \partial D.$$

Further, with (2.5), we derive

$$\begin{aligned}&\frac{c_1\lambda}{\sqrt{\lambda r}}\cdot J_4'(\sqrt{\lambda r})\sin 4t + \frac{c_2\lambda}{\sqrt{\lambda r}}\cdot J_8'(\sqrt{\lambda r})\sin 8t \\ &\quad + 4\left(r+\frac{1}{r}\right)\left[c_1\cdot J_4(\sqrt{\lambda r})\cos 4t\sin 2t + 2c_2\cdot J_8(\sqrt{\lambda r})\cos 8t\sin 2t\right] = 0.\end{aligned}$$

This condition with $t_1 = \pi/6$, $t_2 = \pi/12$ and $r = 1$, can be equivalently written as

$$\begin{cases} \left[\sqrt{\lambda}\cdot J_4'(\sqrt{\lambda}) - 4J_4(\sqrt{\lambda})\right]c_1 + \left[-\sqrt{\lambda}\cdot J_8'(\sqrt{\lambda}) - 8J_8(\sqrt{\lambda})\right]c_2 = 0, \\ \left[\sqrt{3}\cdot\sqrt{\lambda}\cdot J_4'(\sqrt{\lambda}) + 4J_4(\sqrt{\lambda})\right]c_1 + \left[\sqrt{3}\cdot\sqrt{\lambda}\cdot J_8'(\sqrt{\lambda}) - 8\cdot J_8(\sqrt{\lambda})\right]c_2 = 0.\end{cases}$$

Now we are interested in finding nontrivial solutions (c_1, c_2), so the attached determinant must vanishes,

$$\begin{vmatrix} \sqrt{\lambda}\cdot J_4'(\sqrt{\lambda}) - 4J_4(\sqrt{\lambda}) & -\sqrt{\lambda}\cdot J_8'(\sqrt{\lambda}) - 8J_8(\sqrt{\lambda}) \\ \sqrt{3}\cdot\sqrt{\lambda}\cdot J_4'(\sqrt{\lambda}) + 4J_4(\sqrt{\lambda}) & \sqrt{3}\cdot\sqrt{\lambda}\cdot J_8'(\sqrt{\lambda}) - 8\cdot J_8(\sqrt{\lambda})\end{vmatrix} = 0\,.$$

Hence we must have

$$\begin{aligned}&\lambda\sqrt{3}\cdot J_4'(\sqrt{\lambda})J_8'(\sqrt{\lambda}) + 4(\sqrt{3}-1)\sqrt{\lambda}J_4'(\sqrt{\lambda})J_8(\sqrt{\lambda}) \\ &\qquad + 2(1-\sqrt{3})\sqrt{\lambda}J_4(\sqrt{\lambda})J_8'(\sqrt{\lambda}) + 32J_4(\sqrt{\lambda})J_8(\sqrt{\lambda}) = 0.\end{aligned} \tag{2.6}$$

Figure 1: Zoom of the graphic

To solve (2.6) we use the formula see Abramowitz and Stegun

$$J'_m(x) = J_{m-1}(x) - \frac{4}{x} J_m(x) .$$

For the estimation of the approximative solutions we use the assymptotic expansion of the Bessel functions as in Ikonomou, Köhler and Jacob. Let $\nu \in \mathbf{R}$ and let $p \in \mathbf{N}$ such that $\nu - p \leq \frac{1}{2}$. We approximate:

$$J_\nu(x) \simeq J_{\nu,p}(x) ,$$

where

$$J_{\nu,p}(x) = \frac{1}{\sqrt{2\pi x}} \sum_{i=1}^{2} e^{\sigma_k i \varphi_\nu(x)} P_k(x, \nu, p) \quad \text{with} \quad \varphi_\nu(x) = x - \frac{\nu\pi}{2} - \frac{\pi}{4} ,$$

$$P_k(x, \nu, p) = \sum_{m=0}^{m-1} \frac{a_{\nu,m}}{(2\sigma_k i x)^m} \quad \text{with} \quad \sigma_1 = 1, \quad \sigma_2 = -1 ,$$

and

$$a_{\nu,m} = \frac{(1/2 - \nu)_m (1/2 + \nu)_m}{m!} \text{ with } (x)_m = x(x+1)(x+2)\dots(x+m+1) .$$

According to Ikonomou, Köhler and Jacob by denoting

$$\varepsilon(x) = J_\mu(x) J_\nu(x) - J_{\mu,q}(x) J_{\nu,p}(x) , \quad \text{where} \quad \nu - p \leq \frac{1}{2} \quad \text{and} \quad \mu - q \leq \frac{1}{2} ,$$

we have the following estimation:

$$| \varepsilon(x) | = \mathbf{O}(x^{-1-min(p,q)}) \text{ for } x \longrightarrow \infty\,.$$

Also is demonstrated that for $p \geq 6$ and $x \geq 5\pi$ we have $| \varepsilon(x) | \leq 10^{-5}$.

Here are given the roots between 0 and 1000 calculated with a precision of 10^{-4}:

0 35.1878 160.4981 203.7812 252.6651 321.0005 365.7062 455.8866 498.7461 609.7921 651.6186 783.0960 824.2727 975.9514 .

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