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# SMALL-AMPLITUDE HOMOGENISATION OF DIFFUSION EQUATION

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**Abstract:** The small-amplitude homogenisation for the parabolic problem with time dependent periodic coefficients is considered. Calculating the Fourier series we obtain the explicit formula for the leading terms in the expansion of the homogenisation limit, which can be compared to the results obtained by using parabolic variant H-measures.

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**Key Words:** small-amplitude homogenisation, parabolic equation

## 1. Introduction

The small-amplitude homogenisation consists in taking a sequence of coefficients, their difference being proportional to a small parameter, and then computing the first correction on the limit. The explicit formula for the correction in the elliptic case can in general be obtained by using H-measures, a tool introduced around 1990 by Luc Tartar [6] and (independently) Patrick Gérard. On the other hand, in the case of periodic coefficients, one can perform the calculation using Fourier expansions.

In this work, we consider a similar calculation, but for parabolic problems. The homogenisation theory for parabolic problems has been introduced in [5]; for some more recent results see also [7, 3].

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Recently, the first author (jointly with Martin Lazar) introduced several parabolic variants of H-measures [1], which allowed a number of applications to be extended from elliptic to parabolic equations, in particular the small-amplitude homogenisation. In this work we provide a different approach to the periodic framework.

We consider a sequence of parabolic problems in a domain $Q = \langle 0, T\rangle \times \Omega$, where $\Omega \subseteq \mathbf{R}^d$ is an open bounded set:

$$\begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}_n \nabla u) = f \\ \qquad\qquad u_n(0,\cdot) = u_0\ . \end{cases} \tag{1}$$

The coefficients $\mathbf{A}_n \in \mathrm{L}^\infty(Q; \mathrm{M}_{d\times d})$ satisfy the inequalities (valid for any $\boldsymbol{\xi} \in \mathbf{R}^d$ and almost everywhere in $(t, \mathbf{x}) \in Q$):

$$\begin{aligned} \mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} &\geq \alpha|\boldsymbol{\xi}|^2\ , \\ \mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} &\geq \frac{1}{\beta}|\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}|^2\ , \end{aligned} \tag{2}$$

i.e. that they belong to the set $\mathcal{M}(\alpha, \beta; Q)$.

There exists a unique solution $u_n$ of (1) satisfying the homogeneous Dirichlet boundary condition, if $f \in \mathrm{L}^2(0, T; \mathrm{H}^{-1}(\Omega))$, and $u_0 \in \mathrm{L}^2(\Omega)$ (for details see [4]).

We say that a sequence of matrix functions $\mathbf{A}_n \in \mathcal{M}(\alpha, \beta; Q)$ *H-converges* to $\mathbf{A}_\infty \in \mathcal{M}(\alpha', \beta'; Q)$ if for any $f \in \mathrm{L}^2(0, T; \mathrm{H}^{-1}(\Omega))$ and $u_0 \in \mathrm{L}^2(\Omega)$ the solutions $u_n$ of parabolic problems (1) satisfy

$$\begin{aligned} u_n &\longrightarrow u_\infty \quad \text{in } \mathrm{L}^2(0, T; \mathrm{H}^1_0(\Omega))\ , \\ \mathbf{A}_n\nabla u_n &\longrightarrow \mathbf{A}_\infty\nabla u_\infty \quad \text{in } \mathrm{L}^2(Q; \mathbf{R}^d)\ , \end{aligned}$$

implying that $u_\infty$ satisfies (1) with $n = \infty$. Moreover, $\mathbf{A}_\infty \in \mathcal{M}(\alpha, \beta; Q)$.

## 2. Periodic Homogenisation

In the periodic case the explicit formulae for the homogenisation limit are known. We shall first briefly recall the results from [2].

Let $Z$ be the unit periodic cell $[0, 1]^{d+1} = [0, 1] \times Y$ and suppose that $\mathbf{A} \in \mathcal{M}(\alpha, \beta; \mathbf{R}^{d+1})$ is a $Z$-periodic matrix function:

$$\mathbf{A}(\tau, \mathbf{y}) = \mathbf{A}(\tau + l, \mathbf{y} + \mathbf{k})\,, \quad l \in \mathbf{Z}\,,\ \mathbf{k} \in \mathbf{Z}^d,\ \text{a.e. } (\tau, \mathbf{y}) \in Z\,.$$

For given $\rho \in \langle 0, \infty\rangle$ we define the sequence $\mathbf{A}_n$ by

$$\mathbf{A}_n(t,\mathbf{x}) = \mathbf{A}(n^\rho t, n\mathbf{x})\,.$$

Then $\mathbf{A}_n$ $H$-converges to a constant matrix $\mathbf{A}_\infty$ defined by

$$\mathbf{A}_\infty \mathbf{h} = \int_Z \mathbf{A}(\tau,\mathbf{y})(\mathbf{h} + \nabla w(\tau,\mathbf{y}))\,d\tau d\mathbf{y}\,, \tag{3}$$

where, for given $\mathbf{h}$, $w$ is a solution of some boundary value problem, depending on $\rho$:

a) If $\rho \in \langle 0, 2\rangle$ then, for almost every $\tau \in [0,1]$, $w(\tau,\cdot)$ is the unique solution of

$$\begin{aligned} &-\mathsf{div}\,(\mathbf{A}(\tau,\cdot)(\mathbf{h} + \nabla w(\tau,\cdot))) = 0\,,\\ &w(\tau,\cdot) \in \mathrm{H}^1_{\mathrm{per}}(Y)\,,\ \int_Y w(\tau,\mathbf{y})\,d\mathbf{y} = 0\,. \end{aligned} \tag{4}$$

b) If $\rho = 2$, then $w$ is defined as the solution of

$$\begin{aligned} &\partial_t w - \mathsf{div}\,(\mathbf{A}(\mathbf{h} + \nabla w)) = 0\,,\\ &w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}; \mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^d))\,,\ \partial_t w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}; \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d))\,,\\ &w \text{ is } Z\text{-periodic}\,,\ \int_Z w\,d\tau d\mathbf{y} = 0\,. \end{aligned} \tag{5}$$

c) If $\rho > 2$, then we define $\widetilde{\mathbf{A}}(\mathbf{y}) = \int_0^1 \mathbf{A}(\tau,\mathbf{y})\,d\tau$ and $w$ as the solution of

$$\begin{aligned} &-\mathsf{div}\,(\widetilde{\mathbf{A}}(\mathbf{h} + \nabla w)) = 0\\ &w \in \mathrm{H}^1_{\mathrm{per}}(Y)\,,\ \int_Y w\,d\mathbf{y} = 0\,. \end{aligned} \tag{6}$$

**Remark 1.** In the cases (a) and (c) the continuity of $\mathbf{A}$ is needed or in the definition of $H$-convergence one has to consider only $f \in \mathrm{L}^p(0,T;\mathrm{W}^{-1,p}(\Omega))$ and $u_0 \in \mathrm{W}^{1,p}_0(\Omega)$.

## 3. Small-Amplitude Homogenisation

In the small-amplitude homogenisation problem one considers a sequence of small perturbations of a constant coercive matrix $\mathbf{A}_0 \in \mathrm{M}_{d\times d}$:

$$\mathbf{A}^n_\gamma(t,\mathbf{x}) = \mathbf{A}_0 + \gamma \mathbf{B}^n(t,\mathbf{x})\,.$$

Here, we limit our attention to periodic coefficients: $\mathbf{B}^n(t,\mathbf{x}) = \mathbf{B}(n^\rho t, n\mathbf{x})$, where $\mathbf{B}$ is a $Z$-periodic $\mathrm{L}^\infty$ matrix function satisfying $\int_Z \mathbf{B}\,d\tau d\mathbf{y} = 0$. For $\gamma$ small enough, we have the $H$-convergence to a limit depending analytically on

$\gamma$:

$$\mathbf{A}^n_\gamma \xrightarrow{H} \mathbf{A}_\gamma = \mathbf{A}_0 + \gamma\mathbf{B}_0 + \gamma^2\mathbf{C}_0 + o(\gamma^2)\,,$$

and (3) yields a formula for $\mathbf{A}_\gamma$:

$$\begin{aligned}\mathbf{A}_\gamma\mathbf{h} &= \int_Z (\mathbf{A}_0 + \gamma\mathbf{B}))(\mathbf{h} + \nabla w_\gamma)\,d\tau d\mathbf{y}\\ &= \mathbf{A}_0\mathbf{h} + \int_Z \mathbf{A}_0\nabla w_\gamma + \gamma\int_Z \mathbf{B}\mathbf{h} + \gamma\int_Z \mathbf{B}\nabla w_\gamma\\ &= \mathbf{A}_0\mathbf{h} + \gamma\int_Z \mathbf{B}\nabla w_\gamma\,.\end{aligned}$$

In the last equality the second term equals zero by Gauss' Theorem, as $w_\gamma$ is a periodic function. Since $w_\gamma$ is a solution of some (initial–)boundary value problem, depending on $\rho$, it also depends analytically on $\gamma$:

$$w_\gamma = w_0 + \gamma w_1 + o(\gamma)\,. \tag{7}$$

The first order term vanishes, as an easy consequence of the fact that $\mathbf{A}_0$ is constant. Therefore, we have

$$\mathbf{A}_\gamma\mathbf{h} = \mathbf{A}_0\mathbf{h} + \gamma^2\int_Z \mathbf{B}\nabla w_1 + o(\gamma^2)\,,$$

so we conclude that $\mathbf{B}_0 = \mathbf{0}$ and $\mathbf{C}_0\mathbf{h} = \int_Z \mathbf{B}\nabla w_1$. From this formula, by using the Fourier series, one can calculate the second-order approximation $\mathbf{C}_0$. Off course, we must treat separately each one of the above three cases for $\rho$.

a) $\rho \in \langle 0, 2\rangle$. Let us fix $\tau \in [0, 1]$. If we write down the boundary value problem (3) with coefficient $\mathbf{A}_0 + \gamma\mathbf{B}$ instead of $\mathbf{A}$ and insert the expression (0) for $w$, we conclude that $w_1$ solves

$$\begin{aligned}&-\mathsf{div}\,(\mathbf{A}_0\nabla w_1(\tau,\cdot)) = \mathsf{div}\,(\mathbf{B}\mathbf{h})\\ &w_1(\tau,\cdot) \in \mathrm{H}^1_{\mathrm{per}}(Y)\,,\ \int_Y w_1(\tau,\mathbf{y})\,d\mathbf{y} = 0\,.\end{aligned} \tag{8}$$

Because of $\int_Y w_1(\tau,\mathbf{y})\,d\mathbf{y} = 0$, for any $\tau$, the Fourier series of $w_1$ reads $w_1 = \sum_{(l,\mathbf{k})\in J} a_{l\mathbf{k}} e^{2\pi i(l\tau + \mathbf{k}\cdot\mathbf{y})}$, where $J = \mathbf{Z} \times \{\mathbf{Z}^d \setminus \mathbf{0}\}$. Straightforward calculation gives us

$$\begin{aligned}\nabla w_1 &= \sum_J 2\pi i\mathbf{k}\, a_{l\mathbf{k}} e^{2\pi i(l\tau+\mathbf{k}\cdot\mathbf{y})}\,,\\ \mathsf{div}\,\mathbf{A}_0\nabla w_1 &= \sum_J (2\pi i)^2\mathbf{A}_0\mathbf{k}\cdot\mathbf{k}\, a_{l\mathbf{k}} e^{2\pi i(l\tau+\mathbf{k}\cdot\mathbf{y})}\,.\end{aligned}$$

For $\mathbf{B}$, using the notation $I = \mathbf{Z}^{d+1} \setminus \{\mathbf{0}\}$, we get

$$\mathbf{B} = \sum_I \mathbf{B}_{l\mathbf{k}} e^{2\pi i(l\tau + \mathbf{k}\cdot\mathbf{y})}\,,$$

$$\mathsf{div}\,\mathbf{B}\mathbf{h} = \sum_I 2\pi i\,\mathbf{B}_{l\mathbf{k}}\mathbf{h}\cdot\mathbf{k}\,e^{2\pi i(l\tau + \mathbf{k}\cdot\mathbf{y})}\,.$$

Equation in (8) leads to a relation among corresponding Fourier coefficients

$$2\pi i\mathbf{A}_0\mathbf{k}\cdot\mathbf{k}\,a_{l\mathbf{k}} = -\mathbf{B}_{l\mathbf{k}}\mathbf{h}\cdot\mathbf{k}\,,\quad (l,\mathbf{k}) \in \mathbf{Z}^{d+1}\,,$$

which leads to formula

$$a_{l\mathbf{k}} = \begin{cases} -\dfrac{\mathbf{B}_{l\mathbf{k}}\mathbf{h}\cdot\mathbf{k}}{2\pi i\mathbf{A}_0\mathbf{k}\cdot\mathbf{k}}\,, & (l,\mathbf{k}) \in J\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Finally, we obtain

$$\mathbf{C}_0\mathbf{h} = \int_Z \Big(\sum_I \mathbf{B}_{l\mathbf{k}} e^{2\pi i(l\tau + \mathbf{k}\cdot\mathbf{y})}\Big)\Big(\sum_J (2\pi i\mathbf{k}')a_{l'\mathbf{k}'} e^{2\pi i(l'\tau + \mathbf{k}'\cdot\mathbf{y})}\Big)\,d\tau d\mathbf{y}\,.$$

Due to the orthogonality, for the non-vanishing terms in the above product of two series we have $l' = -l$ and $\mathbf{k}' = -\mathbf{k}$. Therefore,

$$\mathbf{C}_0\mathbf{h} = -2\pi i\sum_J \mathbf{B}_{l\mathbf{k}}\mathbf{k}a_{-l,-\mathbf{k}} = -\sum_J \mathbf{B}_{l\mathbf{k}}\mathbf{k}\frac{\mathbf{B}_{-l,-\mathbf{k}}\mathbf{h}\cdot\mathbf{k}}{\mathbf{A}_0\mathbf{k}\cdot\mathbf{k}}\,.$$

Since $\mathbf{B}$ is a real matrix function, we have $\overline{\mathbf{B}_{l\mathbf{k}}} = \mathbf{B}_{-l,-\mathbf{k}}$ which implies formula

$$\mathbf{C}_0 = -\sum_J \frac{\mathbf{B}_{l\mathbf{k}}\mathbf{k}\otimes\mathbf{B}_{l\mathbf{k}}\mathbf{k}}{\mathbf{A}_0\mathbf{k}\cdot\mathbf{k}}\,.$$

b) $\rho = 2$. The calculation is similar to the first case; the only difference being in the equation for $w_1 = \sum_{(l,\mathbf{k})\in I} a_{l\mathbf{k}} e^{2\pi i(l\tau + \mathbf{k}\cdot\mathbf{y})}$:

$$\partial_\tau w_1 - \mathsf{div}\,(\mathbf{A}_0\nabla w_1(\tau,\cdot)) = \mathsf{div}\,(\mathbf{B}\mathbf{h})\,,$$

implying the following relation for the Fourier coefficients

$$(l - 2\pi i\mathbf{A}_0\mathbf{k}\cdot\mathbf{k}a_{l\mathbf{k}}) = \mathbf{B}_{l\mathbf{k}}\mathbf{h}\cdot\mathbf{k}\,,\quad (l,\mathbf{k}) \in I\,,$$

and the formula for second order approximation of the $H$-limit:

$$\mathbf{C}_0 = -\sum_J \frac{\mathbf{B}_{l\mathbf{k}}\mathbf{k}\otimes\mathbf{B}_{l\mathbf{k}}\mathbf{k}}{\frac{l}{2\pi i} + \mathbf{A}_0\mathbf{k}\cdot\mathbf{k}}\,.$$

c) $\rho > 2$. In this case $w_1$ does not depend on $\tau$. After introducing $\widetilde{\mathbf{B}}(\mathbf{y}) := \int_0^1 \mathbf{B}(\tau,\mathbf{y})\,d\tau$ this case actually has the same behaviour as the one in the elliptic setting, giving the formula

$$\mathbf{C}_0 = - \sum_{\mathbf{Z}^d \setminus \{\mathbf{0}\}} \frac{\widetilde{\mathbf{B}}_{\mathbf{k}}\mathbf{k} \otimes \widetilde{\mathbf{B}}_{\mathbf{k}}\mathbf{k}}{\mathbf{A}_0\mathbf{k} \cdot \mathbf{k}} .$$

## References


[1] Nenad Antonić, Martin Lazar, H-measures and variants applied to parabolic equations, *J. Math. Anal. Appl.*, **343** (2008), 207-225.

[2] Alain Bensoussan, Jacques-Louis Lions, George Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland (1978).

[3] Andrea Dall'Aglio, François Murat, A corrector result for H-converging parabolic problems with time-dependent coefficients, *Annali della Scuola Normale Superiore di Pisa*, **25** (1997), 329-373.

[4] Robert Dautray, Jacques-Louis Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, 1-6, Springer (1992).

[5] Sergio Spagnolo, Convergence of parabolic equations, *Bolletino della Unione Matematica Italiana*, **14-B** (1977), 547–568.

[6] Luc Tartar, *Homogenisation, Compensated Compactness, and H-Measures*, Unpublished Lecture Notes.

[7] Vasili V. Žikov, Sergei M. Kozlov, Ol'ga A. Oleinik, O G-shodimosti paraboličeskih operatorov, *Uspehi Matematičeskih Nauk*, **36**, No. 1 (1981), 11-58.