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THE ASYMPTOTIC ANALYSIS OF THE SEQUENCES

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Abstract: We present here the principal problems of the asymptotic analysis of the sequences. These are: to find a simpler principal part for a divergent sequence or series, to obtain the speed of convergence for a convergent sequence or series by the first iterated limit or by a two sided estimation and finally to obtain an asymptotic expansion. Many of our results are remembered here.

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1. Introduction

The asymptotic analysis is considered in relation to a real or complex function defined in a part A of a topological space X (often having $X = \mathbb{R}$ or $X = \mathbb{C}$), to describe the function in a neighbourhood of an accumulation point x_0, using some "simpler" functions. This can be produced in the particular case $A = \mathbb{N}$, $x_0 = \infty$ (the unique accumulation point of $\mathbb{N}$); then the function is a sequence and we obtain the discrete asymptotic analysis.

The general definitions and conventions of the asymptotic analysis remain valid in this case. So, for two sequences $(a_n)_n$ and $(b_n)_n$, we have:

(α) $a_n = O(b_n)$ if there exists two constants $M > 0$ and $n_0 \in \mathbb{N}$ such that $|a_n| \leq M|b_n|$, for any $n \geq n_0$;

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(β) $a_n = o(b_n)$ if $\lim\limits_{n\to\infty} a_n/b_n = 0$;

(γ) $a_n = O(1)$ if $(a_n)_n$ is bounded;

(δ) $a_n = o(1)$ if $\lim\limits_{n\to\infty} a_n = 0$;

(ε) $a_n \sim b_n$ if $\lim\limits_{n\to\infty} a_n/b_n = 1$.

If $(x_n)_n$ is a sequence and $(u_k(n))_k$ is a family of sequences (functions of natural variable n), indexed by $k \in \mathbb{N}$, such that $u_{k+1}(n) = o\left(u_k(n)\right)$ for any $k \in \mathbb{N}$ and $a_k \in \mathbb{R}$ are such that $x_n \sim a_0 u_0(n) + a_1 u_1(n) + \ldots + a_k u_k(n)$, for any $k \in \mathbb{N}$, we say that the series $\sum\limits_{k=0}^{\infty} a_k u_k(n)$ is an asymptotic expansion of x_n. The coefficients $a_0, a_1, a_2, \ldots$ are called the coefficients of the expansion or the iterated limits of x_n (respecting the sequence $(u_k(n))_k$). They are given by the succesive formulas $a_0 = \lim\limits_{n\to\infty} \frac{x_n}{u_0(n)}$, $a_1 = \lim\limits_{n\to\infty} \frac{x_n - a_0 u_0(n)}{u_1(n)}$, $a_2 = \lim\limits_{n\to\infty} \frac{x_n - a_0 u_0(n) - a_1 u_1(n)}{u_2(n)}$, etc. The sum $\sum\limits_{k=0}^{m} a_k u_k(n) + O\left(u_{m+1}(n)\right)$ is the expansion of order m of x_n.

In the asymptotic analysis, the principal problems related to a sequence can be considered the following:

(a) to obtain the order of magnitude (in terms of "simpler" sequences);

(b) to obtain the convergence of the given sequence or (if this is not convergent) to obtain the convergence of a related sequence;

(c) to obtain the first iterated limit (respecting an auxiliary scale of sequences);

(c′) to obtain a two sided estimation of the speed of convergence; this is closely related to (c) because it permits to obtain again the limit of (c);

(d) to obtain an asymptotic expansion (respecting a given scale of sequences);

In the following we will give several examples for any of these aims.

2. Obtaining the Order of Magnitude

For the harmonic sum $H_n = 1 + 1/2 + \ldots + 1/n$, the order of magnitude is given by *Euler* $H_n = \ln n + \gamma + \varepsilon_n$, where $\gamma = C = \lim\limits_{n\to\infty} (H_n - \ln n) = 0,577\ldots$ is its constant and $\varepsilon_n = o(1)$.

The factorial's magnitude is given by the *Stirling*'s formula $n! \approx n^n \mathrm{e}^{-n}\sqrt{2\pi n}$, which has the precise semnification that

$$\lim_{n\to\infty}\frac{n!}{n^n \mathrm{e}^{-n}\sqrt{2\pi n}}=1.$$

The number of primes not exceeding a given natural number n is denoted by $\pi(n)$ and we have $\pi(n)\approx n/\ln n$, that means

$$\lim_{n\to\infty}\frac{\pi(n)}{\dfrac{n}{\ln n}}=1.$$

For the "accelerated" factorial $\overline{n}=1^1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n$, *E. Cesàro* has established the formula$\overline{n}^{\frac{1}{n}}\approx \mathrm{e}^{\frac{n}{4}}n^{\frac{-(n+1)}{2}}$, i.e., $\lim\limits_{n\to\infty}\overline{n}^{\frac{1}{n}}\Big/\mathrm{e}^{\frac{n}{4}}n^{\frac{-(n+1)}{2}}=1$ (see Polya [5]). Later *T. Popoviciu* asked to examine the "decelerated" factorial $\underline{n}=1^n 2^{n-1}3^{n-2}\cdot\ldots\cdot n^1$, a problem solved by *A. Lupaş* and *L. Lupaş*

$$\overline{n}=C\cdot n^{\frac{n(n+1)}{2}+\frac{1}{12}}\exp\left(-\frac{n^2}{4}+\frac{1}{720n^2}-\frac{\theta_n}{5040n^4}\right),$$

with $\theta_n\in(0,1)$, where $C=\mathrm{e}^{-\frac{5}{36}}\pi^{\frac{1}{6}}\exp\left(\frac{2}{3}\int\limits_0^{\frac{1}{2}}\ln\Gamma(t)\mathrm{d}t\right)$ (see Lupaş [2]).

Also, *A. Lupaş* and *L. Lupaş* obtained, with $w_n\in(0,1)$

$$\frac{\overline{n}}{\underline{n}}=C^2(2\pi)^{-\frac{(n+1)}{2}}\cdot n^{-\frac{n}{2}-\frac{1}{2}}\exp\left(\frac{n^2}{2}+n-\frac{1}{12n}+\frac{1}{180n^2}+\frac{w_n}{360n^3}\right).$$

Let now consider the regular lacunary harmonic sums

$$H_{n,\alpha}^{(r)}\overset{\text{def}}{=\!=}\frac{1}{\alpha}+\frac{1}{\alpha+r}+\frac{1}{\alpha+2r}+\ldots+\frac{1}{\alpha+(n-1)r},$$

where $r\in\mathbb{N}^*$, $r>1$, $1\le\alpha\le r$. The order of magnitude is given in Vernescu [11]; we have , for $r=2,3$ and 4:

$$\begin{cases} H_{n,1}^{(2)}=1+\dfrac{1}{3}+\dfrac{1}{5}+\ldots+\dfrac{1}{2n-1}=\dfrac{1}{2}\ln n+\dfrac{1}{2}(\gamma+2\ln 2)+o(1);\\ H_{n,2}^{(2)}=\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{6}+\ldots+\dfrac{1}{2n}=\dfrac{1}{2}\ln 2+\dfrac{1}{2}\gamma+o(1);\end{cases}$$

$$\begin{cases} H_{n,1}^{(3)}=1+\dfrac{1}{4}+\ldots+\dfrac{1}{3n-2}=\dfrac{1}{3}\ln n+\dfrac{1}{3}\left(\gamma+\dfrac{3}{2}\ln 3+\dfrac{\pi\sqrt{3}}{6}\right)+o(1);\\ H_{n,2}^{(3)}=\dfrac{1}{2}+\dfrac{1}{5}+\ldots+\dfrac{1}{3n-1}=\dfrac{1}{3}\ln n+\dfrac{1}{3}\left(\gamma+\dfrac{3}{2}\ln 3-\dfrac{\pi\sqrt{3}}{6}\right)+o(1);\\ H_{n,3}^{(3)}=\dfrac{1}{3}+\dfrac{1}{6}+\ldots+\dfrac{1}{3n}=\dfrac{1}{3}\ln n+\dfrac{1}{3}\gamma++o(1);\end{cases}$$

$$\begin{cases} H_{n,1}^{(4)} = 1 + \frac{1}{5} + \frac{1}{9} + \ldots + \frac{1}{4n-3} = \frac{1}{4}\ln n + \frac{1}{4}\left(\gamma + 3\ln 2 + \frac{\pi}{2}\right) + o(1); \\ H_{n,2}^{(4)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \ldots + \frac{1}{4n-2} = \frac{1}{4}\ln n + \frac{1}{4}(\gamma + 2\ln 2) + o(1); \\ H_{n,3}^{(4)} = \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \ldots + \frac{1}{4n-1} = \frac{1}{4}\ln n + \frac{1}{4}\left(\gamma + 3\ln 2 - \frac{\pi}{2}\right) + o(1); \\ H_{n,4}^{(4)} = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \ldots + \frac{1}{4n} = \frac{1}{4}\ln n + \frac{1}{4}\gamma + o(1). \end{cases}$$

For the sum $S_n = \log_2 3 + \log_3 4 + \ldots + \log_n(n+1)$, introduced by *L. Panaitopol*, the order of magnitude is $S_n = (n-1) + \ln(\ln n) + A + o(1)$, where

$$A = \gamma + \lim_{n\to\infty}\left(\sum_{k=2}^{n}\frac{1}{k\ln k} - \ln(\ln n)\right) \qquad \text{(see Vernescu [12]).}$$

Let $\Omega_n = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n}$ be; the order of magnitude of this sequence (which tends to 0) is $\Omega_n \approx \frac{1}{\sqrt{\pi n}}$, as it results from an inequality called in the monograph *Mitrinović* and *Vasić* [3] the inequality of *Wallis*

$$\frac{1}{\sqrt{\pi\,(n+1/2)}} < \Omega_n < \frac{1}{\sqrt{\pi n}}.$$

Consider now, for $r \in \mathbb{N}$, $r > 1$ and $\alpha, \beta \in \mathbb{R}$, $0 < \alpha < \beta \le r$, the sequence

$$\Omega_{n,r}^{\frac{\alpha}{\beta}} = \frac{\alpha(\alpha+r)(\alpha+2r)\cdot\ldots\cdot(\alpha+(n-1)r)}{\beta(\beta+r)(\beta+2r)\cdot\ldots\cdot(\beta+(n-1)r)}.$$

In Vernescu [15] we have obtained the order of magnitude of this sequence, namely

$$\Omega_{n,r}^{\frac{\alpha}{\beta}} \sim \frac{\Gamma\left(\frac{\beta}{r}\right)}{\Gamma\left(\frac{\alpha}{r}\right)} \cdot \frac{1}{n^{\frac{\beta-\alpha}{r}}}.$$

3. Obtaining the Convergence of a Given Sequence or (If it is Not Convergent) Obtaining the One of a Related Sequence

A classic example is the one of the harmonic sum H_n; it does not converges, but the related sequence of general term $c_n = H_n - \ln n$ is convergent; this has entirely solved by *Euler*. Analogously, if $T_n = \sum_{k=2}^{n}\frac{1}{k\ln k}$, this tends to ∞, but the sequence of general term $a_n = T_n - \ln(\ln n)$ is convergent.

Moreover, the sum $S_n = \log_2 3 + \log_3 4 + \ldots + \log_n(n+1)$ is divergent, but

(2.7) shows us that the sequence of general term $x_n = S_n - (n-1) - \ln(\ln n)$ converges.

The formulas of $H^{(r)}_{n,\alpha}$ also give, each of them, a convergent sequence.

4. Obtaining the First Iterated Limit (Respecting an Auxiliary Scale of Sequences)

To obtain a "measure" of the speed of convergence, the first iterated limit of a sequence can be useful. A classic example is the following

$$\lim_{n\to\infty} n\left(\mathrm{e} - \left(1+\frac{1}{n}\right)^n\right) = \frac{\mathrm{e}}{2} \quad \text{(see Polya [5], Vernescu [9])},$$

$$\lim_{n\to\infty} n!n\left(\mathrm{e} - \left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right)\right) = 1,$$

that shows us an enormous difference between the speed of convergence of these sequences. We also cite the following iterated limits

$$\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^{n+1} - \mathrm{e}\right) = \frac{\mathrm{e}}{2} \qquad \text{(see Vernescu [7])}$$

$$\lim_{n\to\infty} n\left(\frac{1}{\mathrm{e}} - \left(1-\frac{1}{n}\right)^n\right) = \frac{1}{2\mathrm{e}} \qquad \text{(see Niculescu [4])}$$

$$\lim_{n\to\infty} n\left(\left(1-\frac{1}{n}\right)^{n-1} - \frac{1}{\mathrm{e}}\right) = \frac{1}{2\mathrm{e}} \qquad \text{(see Niculescu [4])}$$

$$\lim_{n\to\infty} n\left(c_n - C\right) = \frac{1}{2} \qquad \text{(see Vernescu [9])}$$

$$\lim_{n\to\infty} n\left(x^{(r)}_{n,\alpha} - x^{(r)}_{\alpha}\right) = \frac{|r-2\alpha|}{2r^2} \qquad \text{(see Vernescu [7])}$$

where $x^{(r)}_{n,\alpha} = H^{(r)}_{n,\alpha} - \frac{1}{r}\ln n$ and $x^{(r)}_{\alpha} = \lim\limits_{n\to\infty} x^{(r)}_{n,\alpha}$. Many other examples also can be considered.

5. Obtaining a Two Sided Estimation of the Speed of Convergence

We also have many examples. Consider some of these

$$\frac{\mathrm{e}}{2n+2} < \mathrm{e} - \left(1+\frac{1}{n}\right)^n < \frac{\mathrm{e}}{2n+1} \qquad \text{(see Polya [5], Vernescu [9])}$$

$$\frac{\mathrm{e}}{2n+1} < \left(1+\frac{1}{n}\right)^{n+1} - \mathrm{e} < \frac{\mathrm{e}}{2n} \qquad \text{(see Vernescu [7])}$$

$$\frac{1}{2n\mathrm{e}} < \frac{1}{\mathrm{e}} - \left(1-\frac{1}{n}\right)^n < \frac{1}{(2n-1)\mathrm{e}} \qquad \text{(see Niculescu [4])}$$

$$\frac{1}{(2n-1)\mathrm{e}} < \left(1-\frac{1}{n}\right)^{n-1} - \frac{1}{\mathrm{e}} < \frac{1}{(2n-2)\mathrm{e}} \qquad \text{(see Niculescu [4])}$$

$$\frac{1}{2n+1} < c_n - C < \frac{1}{2n} \qquad \text{(see Vernescu [8]).}$$

Each of these two sided estimations permits to obtain the corresponding iterated limit; but the iterated limit also can be obtained by passing to the real variable and using the methods of calculus.

6. Obtaining the Asymtotic Expansion

An asymptotic expansion gives a deeper characterization of a given sequence, containing all the iterated limits. The one of $n!$ and H_n are famous

$$\ln n! = \left(n+\frac{1}{2}\right)\ln n - n - A + \sum_{k=1}^{m} \frac{(-1)^k B_k}{k(k-1)n^{k-1}} + O\left(\frac{1}{n^{m+1}}\right)$$

$$H_n = \ln n + C + \frac{1}{2n} + \sum_{k=2}^{m} \frac{(-1)^{k-1} B_k}{kn^k} + O\left(\frac{1}{n^{m+1}}\right),$$

where $A = \ln\sqrt{2\pi}$ and B_k are the numbers of *Bernoulli*. The asymptotic expansion for W_n of *Wallis*'s formula (see Toth [6]) is

$$W_n = \frac{\pi}{2}\left(1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \frac{83}{2048n^4} - \frac{143}{8192n^5} + \dots\right).$$

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