International Journal of Pure and Applied Mathematics

Volume 50 No. 2 2009, 283-288

ON A CLASS OF LINEAR MATERIAL LAWS IN CLASSICAL MATHEMATICAL PHYSICS

Rainer Picard

Institut für Analysis
Fachrichtung Mathematik
Fakultät Mathematik und Naturwissenschaften
Technische Universität Dresden
Dresden, 01062, GERMANY
e-mail: rainer.picard@tu-dresden.de

Abstract: A class of linear material laws is considered, which covers a number of diverse initial boundary value problems of classical mathematical physics. The claim that this class is indeed to a large extent sufficiently general can be exemplified for a number of specific models from classical physics. We illustrate the concept for some well-known linear material laws in electrodynamics.

AMS Subject Classification: 37L05, 35F10, 58D25
Key Words: material laws, electrodynamics, evolution equations, classical mathematical physics

1. Introduction

On closer inspection of initial boundary value problems of mathematical physics, in particular those describing wave propagation phenomena one is inclined to describe their general form as

$$\partial_0 V + AU = f \text{ on } \mathbb{R}_{>0},$$
$$V(0+) = \Phi,$$

where A is skew-selfadjoint in a suitable Hilbert space setting. We shall indeed prefer to consider this problem on the whole real time-line and to by-pass the full construction of associated Sobolev lattices, see Picard [2], we shall assume – without loss of generality – that $\Phi = 0$. This turns our problem into

Received: August 14, 2008

$$\partial_0 V + AU = f \text{ on } \mathbb{R}. \tag{1}$$

This evolutionary problem is now completed by an additional rule frequently referred to as a "material law", which for simplicity we assume to be time-translation-invariant and more precisely of the form

$$V = M\left(\partial_0^{-1}\right) U, \tag{2}$$

where $z \mapsto M(z)$ is bounded-operator-valued and analytic in an open ball $B_{\mathbb{C}}(r, r)$ with some positive radius r centered at r. Indeed, to by-pass the construction of associated Sobolev lattices altogether we consider the problem in the compact form

$$\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right) U = f$$

as an operator equation in a suitable Hilbert space setting.

2. The Time Derivative

It is well-known that $\frac{1}{\mathrm{i}}\,\partial_0$ can be established as a selfadjoint operator in the space $L^2(\mathbb{R})$ of equivalence classes of square-integrable complex-valued functions on $\mathbb{R}$. The space $\mathring{C}_\infty(\mathbb{R})$ of smooth complex-valued functions with compact support is densely embedded in the domain. Indeed, this case is occasionally used as a simple example for an explicit spectral representation, which here is provided by the Fourier transform $\mathcal{F}$ given as the unitary extension of

$$\mathring{C}_\infty(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \varphi \mapsto \widehat{\varphi}$$

with

$$\widehat{\varphi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\mathrm{i}\, x\, t)\ \varphi(t)\ dt,\ x \in \mathbb{R}.$$

As a spectral representation the Fourier transform makes $\frac{1}{\mathrm{i}}\,\partial_0$ unitarily equivalent to the multiplication by the argument operator m given by $(m\varphi)(x) = x\varphi(x)$ for $x \in \mathbb{R}$ and $\varphi \in \mathring{C}_\infty(\mathbb{R})$:

$$\frac{1}{\mathrm{i}}\,\partial_0 = \mathcal{F}^* m\, \mathcal{F}.$$

Following Picard [1] Section 1.2 we introduce an exponential weight function $t \mapsto \exp(-\nu\, t)$, $\nu \in \mathbb{R}$, and consider the weighted L^2-space $H_{\nu,0}$ generated by completion of $\mathring{C}_\infty(\mathbb{R})$ with respect to the inner product $\langle\,\cdot\,|\,\cdot\,\rangle_{\nu,0}$ given by

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \varphi(t)^*\, \psi(t)\ \exp(-2\nu t)\, dt.$$

The associated norm will be denoted by $|\cdot|_{\nu,0}$. The multiplication operator

$$\begin{aligned}\mathring{C}_\infty(\mathbb{R}) \subseteq H_{\nu,0} &\to \mathring{C}_\infty(\mathbb{R}) \subseteq H_{0,0} = L^2(\mathbb{R})\\ \varphi &\mapsto \exp(-\nu m)\varphi\end{aligned}$$

with

$$(\exp(-\nu m)\varphi)(x) = \exp(-\nu x)\varphi(x),\ x \in \mathbb{R},$$

clearly has a unitary extension, which we shall denote by $\exp(-\nu m)$, where the m serves as a reminder for "multiplication-by-argument". Its inverse will be denoted by $\exp(\nu m)$. Thus, the operator

$$\frac{1}{\mathrm{i}}\,\partial_\nu := \exp(\nu m)\,\frac{1}{\mathrm{i}}\,\partial_0 \exp(-\nu m)$$

defines a unitarily equivalent operator $\frac{1}{\mathrm{i}}\,\partial_\nu$, which is now selfadjoint in $H_{\nu,0}$.

We shall use again the notation ∂_0 for the normal operator $\partial_\nu + \nu$, which is justified since indeed

$$(\partial_\nu + \nu)\varphi = \partial_0\varphi$$

for $\varphi \in \mathring{C}_\infty(\mathbb{R})$.

Obviously we have that the spectrum of ∂_ν is purely imaginary. In fact, the spectrum $\sigma(\partial_\nu)$ is also purely continuous spectrum:

$$\sigma(\partial_\nu) = \sigma_c(\partial_\nu) = \mathrm{i}\,\mathbb{R}.$$

Thus, in particular for $\nu \in \mathbb{R}\setminus\{0\}$ we have the bounded invertibility of $\partial_0 = \partial_\nu + \nu$. With $\mathcal{L}_\nu := \mathcal{F}\exp(-\nu m)$, $\frac{1}{\mathrm{i}}\,\partial_\nu = \mathcal{L}_\nu^* m\,\mathcal{L}_\nu$.

3. Evolutionary Dynamics and Material Laws

We shall now consider the initially stated evolutionary problem in precise terms. For this, we need to extend the operators ∂_0, A to the tensor product spaces $H_{\nu,0}\otimes H$ by interpreting ∂_0 as $\partial_0 \otimes 1_H$ and A as $1_{H_{\nu,0}}\otimes A$ with $1_H : H \to H$, $1_{H_{\nu,0}} : H_{\nu,0} \to H_{\nu,0}$ as identity operators in H, $H_{\nu,0}$, respectively.

In this sense, our aim is to be able to find $U \in H_{\nu,0}\otimes H$ such that for a given $f \in H_{\nu,0}\otimes H$ we have

$$\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)U = f. \tag{3}$$

Here $(M(z))_{z\in B_{\mathbb{C}}(r,r)}$ is a uniformly bounded, holomorphic family of linear operators in H with $r \leq \frac{1}{2\nu}$. It is

$$M\left(\partial_0^{-1}\right) := \mathcal{L}_\nu^*\, M\left(\frac{1}{\mathrm{i}\,m+\nu}\right)\,\mathcal{L}_\nu.$$

Note that $\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)$ abbreviates here *the closure* of the operator $\partial_0 M\left(\partial_0^{-1}\right)+A$ initially considered on its natural domain $D\left(\partial_0\right)\cap D\left(A\right)$. To warrant a solution theory we require an additional constraint on such causal materials:

$$\bigvee_{c\in\mathbb{R}_{>0}}\ \bigwedge_{z\in B_{\mathbb{C}}(r,r)} \mathfrak{Re}\left(z^{-1}M\left(z\right)\right)\geq c. \tag{posdef}$$

Theorem. (Solution Theory) *Let $\left(M\left(z\right)\right)_{z\in B_{\mathbb{C}}(r,r)}$ be a holomorphic family of uniformly bounded linear operators on H, $r\in\mathbb{R}_{>0}$, satisfying our definiteness condition (posdef) and A skew-selfadjoint in H, then we have for $\nu\geq\frac{1}{2r}$ and every $f\in H_{\nu,0}\otimes H$ a unique solution $U\in H_{\nu,0}\otimes H$ of the problem*

$$\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)U=f\,.$$

Moreover, the solution depends continuously on the data in $H_{\nu,0}\otimes H$.

Proof. Continuous invertibility of $\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)$ follows from (posdef). Density of the range follows, since also $\left(\partial_0 M\left(\partial_0^{-1}\right)\right)^*$ satisfies (posdef) and therefore $\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)^*$ only has a trivial null space. A more detailed argument is given in Picard [3]. □

Remark. The solution is also causal in the sense that the solution does not start to be non-zero before the right-hand side does. Although crucial for this approach a more precise statement of this property cannot be supplied here.

In order to demonstrate that this surprisingly simple scheme is indeed capable of describing most of the standard initial boundary value problems of mathematical physics one would have bring them into the form described. Here we have only space for one application to illustrate what is needed to perform this step. We shall investigate the equations of electrodynamics in the light of our theoretical considerations above.

4. An Illustration: The Equations of Electrodynamics

To keep matters elementary we take $\mathring{\mathrm{curl}}$ as the closure of the classical vector analytical operation curl if applied to smooth vector fields with compact support in a non-empty, open subset $\Omega\subseteq\mathbb{R}^3$ as a mapping in $L^2\left(\Omega\right)\oplus L^2\left(\Omega\right)\oplus L^2\left(\Omega\right)$. Being in the domain of $\mathring{\mathrm{curl}}$ encodes a generalization of the vanishing of tangential components on the boundary of Ω. With curl denoting the adjoint of $\mathring{\mathrm{curl}}$ we can formulate Maxwell's equations, which are the governing equations of electrodynamics, as

$$\partial_0 \begin{pmatrix} D \\ B \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -J - J_c \\ 0 \end{pmatrix}.$$

Here E denotes the electric and H the magnetic field, whereas D is known as the displacement current density and B as the magnetic induction.

The intrinsic current density term J_c only occurs if the material is conducting. Ohm's law states that there is a linear connection between J_c and the electric field E:

$$J_c = \sigma E$$

with σ bounded, selfadjoint and non-negative mapping in $L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega)$. The system is completed by the linear material relation

$$\begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}$$

with ε, μ as bounded, selfadjoint, strictly positive definite mappings in $L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega)$.

This is not quite the standard form we formulated in the general theory, but it is not hard to reformulate this as

$$\left(\partial_0 \begin{pmatrix} \varepsilon + \sigma\, \partial_0^{-1} & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix}\right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -J \\ 0 \end{pmatrix}$$

which is indeed of our general form.

It is common to also consider in this context the slightly degenerate case, where $\varepsilon = 0$ but σ is selfadjoint, strictly positive-definite and bounded. This problem is frequently referred to as the *eddy current problem* and is also covered by our framework, despite the fact that the problem becomes non-reversible in this case. It may be interesting to inspect also a more complicated material model. We choose Drude-Lorentz type models to show how these can be implemented in our framework.

In generalization of material laws suggested by Drude and Lorentz, see Seeliger [4], we arrive at material laws of the form

$$\begin{pmatrix} D \\ B \end{pmatrix}$$
$$= \begin{pmatrix} \varepsilon + \sum_{k=0}^{N} \left(\alpha_k \left(\kappa_k + \partial_0\right)^{-1} + \beta_k \left(\kappa_{k0} + \partial_0^2 - \kappa_{k1}\partial_0\right)^{-1}\right) & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} + \partial_0^{-1} \begin{pmatrix} \sum_{k=0}^{N} \alpha_k \left(\kappa_k \partial_0^{-1} + 1\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}$$

$$+ \partial_0^{-2} \begin{pmatrix} \sum_{k=0}^{N} \beta_k \left(\left(\kappa_{k0} \partial_0^{-1} - \kappa_{k1}\right) \partial_0^{-1} + 1\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix},$$

where all coefficients κ_k, κ_{0k}, κ_{1k}, $k = 0, \ldots, N$, are originally real numbers but in the light of the above are now allowed to be bounded, linear mappings in $L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega)$, $N \in \mathbb{N}$.

Noting that by choosing $\nu \in \mathbb{R}_{>0}$ sufficiently large, we have a convergent Neumann series expansion

$$\left(\eta_\nu \partial_0^{-1} + 1\right)^{-1} = \sum_{s=0}^{\infty} \left(\eta_\nu \partial_0^{-1}\right)^s$$

for any family $(\eta_\nu)_\nu$ of linear mappings in $H_{\nu,0} \otimes \left(L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega)\right)$ uniformly bounded w.r.t. $\nu \in \mathbb{R}_{>0}$ sufficiently large, we also see that the term $\sum_{k=0}^{N} \alpha_k$ acts as a conductivity term.

Thus, our framework has been exemplified by considering some materials in electrodynamics. It works like-wise for equations of heat transport, the equations of visco-elastic solids as well as for various coupled systems of more complex form, just to mention a few. The full power of the described framework will, however, be demonstrated elsewhere.

References

[1] R. Picard, Hilbert space approach to some classical transforms, *Pitman Research Notes in Mathematics*, Series 196, Harlow, Longman Scientific and Technical; New York, John Wiley and Sons (1989).

[2] R. Picard, Evolution equations as operator equations in lattices of Hilbert spaces, *Glas. Mat.*, III. Ser., **35**, No. 1 (2000), 111-136.

[3] R. Picard, A structural observation for linear material laws in classical mathematical physics, *Preprint MATH-AN-02-2008*, TU Dresden, Germany (2008), To Appear.

[4] R. Seeliger, Elektronentheorie der Metalle, *Encykl. D. Math. Wiss.*, **20** (1922), 777-878.