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PROBLEMS OF STATE ESTIMATION FOR DYNAMICAL CONTROL SYSTEMS WITH QUADRATIC NONLINEARITY

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Abstract: The paper is devoted to the state estimation problems for control systems described by nonlinear differential equations with quadratic nonlinearity. The initial state is assumed to be unknown but bounded with given bounds. We present the ellipsoidal techniques which provide the ellipsoidal estimates of reachable sets of this nonlinear control problem.

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1. Introduction

The paper deals with the problems of state estimation for nonlinear dynamical control system

$$\dot{x} = Ax + \tilde{f}(x)d + G(t)u(t), \quad t_0 \le t \le T, \tag{1}$$

with unknown but bounded initial condition [4, 5, 1, 6]

$$x(t_0) = x_0, \quad x_0 \in X_0 = E(a_0, Q_0) \subset R^n. \tag{2}$$

Here we denote by $E(y, Y)$ the ellipsoid in R^n

$$E(y, Y) = \{x \in R^n : (Y^{-1}(x - y), (x - y)) \le 1\},$$

with center $a \in R^n$ and with symmetric positively definite $n \times n$–matrix Y.

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The nonlinear n–vector function $\tilde{f}(x)$ in (1) is assumed to be of quadratic type

$$\tilde{f}(x) = x'Bx, \tag{3}$$

where B is a symmetrical and positive definite $n \times n$–matrix.

Here the matrix $G(t)$ (of dimension $n \times m$) in (1) is assumed to be continuous on $[t_0, T]$, A is the constant $n \times n$–matrix, $d \in R^n$.

The control $u(t)$ in (1) is a measurable m–vector function with the following constraint. We will assume that $u(\cdot) \in \mathcal{U}$, where

$$\mathcal{U} = \{u(\cdot) : G(t)u(t) \in E(\check{a}, \check{Q}) \subset R^n \text{ a.e. on } [t_0, T]\}. \tag{4}$$

Denote by $x(\cdot) = x(\cdot, t_0, x_0, u(\cdot))$ a solution of system (1)–(4) for some $x_0 \in X_0$ and $u(\cdot) \in \mathcal{U}$. We may assume that all solutions $x(\cdot) = x(\cdot, t_0, x_0, u(\cdot))$ of the system (1) exist on the interval $[t_0, T]$ for any x_0 and $u(\cdot)$ satisfying (2), (4) (results that guarantee this assumption are given in [2]) and belong to a bounded domain $D = \{x \in R^n : \| x \| \leq K\}$ (the existence of such constant $K > 0$ follows from classical theorems of the theory of differential equations and differential inclusions).

Definition. (see [4]) The set-valued map

$$X(\cdot; t_0, X_0) = \bigcup_{x_0 \in X_0,\ u(\cdot) \in \mathcal{U}} \{x(\cdot; t_0, x_0, u(\cdot))\}$$

is called the trajectory tube of the system (1) with the initial state $\{t_0, X_0\}$. The cross-section $X(t) = X(t; t_0, X_0)$ of the trajectory tube $X(\cdot; t_0, X_0)$ at $t \geq t_0$ is the reachable set of the system (1) at the instant $t \in [t_0, T]$ with the initial state $\{t_0, X_0\}$.

The aim of this paper is to present the algorithm of set-valued approximations of trajectory tubes $X(\cdot; t_0, X_0)$ and reachable sets $X(t; t_0, X_0)$ of the system (1)–(4).

The techniques used in this paper are based on the theory of ellipsoidal estimation developed in [5], [1] for linear control system under uncertainty and on the theory of funnel equations for differential inclusions [6].

Basing on results of [2] we present the new algorithm which allows to construct non-convex estimates of reachable sets by means of the union of two special ellipsoidal tubes with precise description of their parameters.

2. Results: Ellipsoidal Estimates of Reachable Sets

Two following auxiliary problems will be used in formulating the results. Let us denote by $B(c,R)$ the ball in R^n

$$B(c,R)=\{x\in R^n : (x-c,x-c)\leq r\},$$

with center $c\in R^n$ and with radius r and

$$\tilde{R}^n_+(k)=\big\{x\in R^n : x_k\geq 0\ \big\},\ \tilde{R}^n_-(k)=\big\{x\in R^n : x_k\leq 0\ \big\}.$$

Problem 1. The matrix $Q=$diag$\big\{q_1^2,...,q_n^2\big\},\ q_i>0\ (i=1,...,n),\ q_k=\min\limits_{i=1,...,n}\{q_i\}$ and the sets $\tilde{S}=B(0,1)\cap\tilde{R}^n_+(k),\ \ \tilde{\tilde{S}}=B(0,1)\cap\tilde{R}^n_-(k)$ are given. Parameters $a_*^{(1)},a_*^{(2)},k_*$ of ellipsoids with minimal volume such that

$$\tilde{S}\subseteq E(a_*^{(1)},k_*^2Q),\quad \tilde{\tilde{S}}\subseteq E(a_*^{(2)},k_*^2Q) \tag{5}$$

are required to find.

Lemma 1. *Parameters of ellipsoids (5) are defined by formulas*

$$a_*^{(1)}=\big(0,0,...0,a_k,0,...,0\big)\in R^n,$$

$$a_*^{(2)}=\big(0,0,...0,-a_k,0,...,0\big)\in R^n,\ a_k=\frac{1}{2}\Big(1-\frac{q_k^2}{q_j^2}\Big),$$

$$k_*=\frac{1}{2q_k}\Big(1+\frac{q_k^2}{q_j^2}\Big),\ q_j=\min_{i=1...n,\ i\neq k}\{q_i\}.$$

Problem 2. For two unit balls $B(c,1),\ B(-c,1)\subset R^n$ with $c=(c_0,0,...,0)\in R^n$ we need to find the ellipsoid $E(a_*,Q_*)$ having minimal volume and such that the following inclusion is true

$$B(c,1)\cup B(-c,1)\subseteq E(a_*,Q_*). \tag{6}$$

Lemma 2. *The parameters a_*,Q_* of ellipsoid (6) are defined by formulas:*

$$a_*=(0,0,...,0),\quad Q_*=\mathrm{diag}\big\{a_1^2,a_2^2,...,a_2^2\big\},$$

where

$$a_1^2=(c_0-1)^2+\frac{c_0}{2bn}(2n+b)^2,\quad a_2^2=1+\frac{c_0b}{2n},$$
$$b=c_0(1-n)+\sqrt{c_0^2(1-n)^2+4n}.$$

Proof. The proofs of the Lemmas 1, 2 were presented in [3].

Now we consider the main result of the paper which is the iterative algorithm of ellipsoidal estimating of reachable sets of system (1)–(4). The algorithm consists of repeated iterations and each iteration uses four basic steps.

Step 1. Given some initial ellipsoid, we cover it by the union of two special ellipsoids. The step is based on the solution of Problem 1 where the matrix Q is found from the data of nonlinear system (1)–(4). So at the end of this step we obtain two new ellipsoids.

Step 2. Every ellipsoid obtained at the previous step we consider as the initial set for the system (1) and find the ellipsoidal estimate of related reachable set (as was explained in [2]). So at the end of this step we will have again two new ellipsoids.

Step 3. We find the optimal ellipsoid which estimates the union of two last ellipsoids. The step is based on the solution of Problem 2 where the balls $B(-c,1)$, $B(c,1)$ are constructed from the resulting ellipsoids of step 2 by appropriate space transformation. So at the end of this step we obtain the new ellipsoid.

Step 4. We assume that the received ellipsoid is a new initial set and return at the first step.

The main idea of the above algorithm is based on the inclusion

$$X(t_0+\sigma;t_0,X_0) \subseteq \bigcup_{x\in X_0} \big\{x+\sigma(Ax+f(x)d+E(\breve{a},\breve{Q}))$$
$$+o(\sigma)B(0,1)\big\} \subseteq E(a(\sigma),Q(\sigma))+\sigma E(\breve{a},\breve{Q})+o(\sigma)B(0,1)$$
$$\lim_{\sigma\to+0}\sigma^{-1}o(\sigma)=0, \quad a(\sigma)=\sigma d, \quad Q(\sigma)=k^2(I+\sigma A)B^{-1}(I+\sigma A)'$$

which follows from the funnel equation [4], [6] and is true if $X_0=E(0,B^{-1})$ [2]. So at each iteration (Steps 1-4) instead of X_0 we use its upper estimate generated by one special ellipsoid as in [2] or by the union of two special ellipsoids as is done here.

3. Example and Concluding Remarks

The following example illustrates the algorithm.

Example. Consider the following nonlinear control system

$$\begin{cases} \dot{x}_1 &= -\,x_1+u_1, \\ \dot{x}_2 &= 0.5x_2+3(0.25x_1^2+x_2^2)+u_2, \end{cases} \qquad t_0\le t\le T. \tag{7}$$

Here we take $t_0=0$, $T=0.02$ and we use $h=0.01$ as the step of the iteration process. We assume that $a_0=(0,1)$, $\breve{a}=(0,0)$ and

$$X_0=E(a_0,Q_0),\ Q_0=\begin{pmatrix} 2 & 0\\ 0 & 3\end{pmatrix},\ \breve{Q}=\begin{pmatrix} 0.1 & 0\\ 0 & 0.1\end{pmatrix}. \tag{8}$$

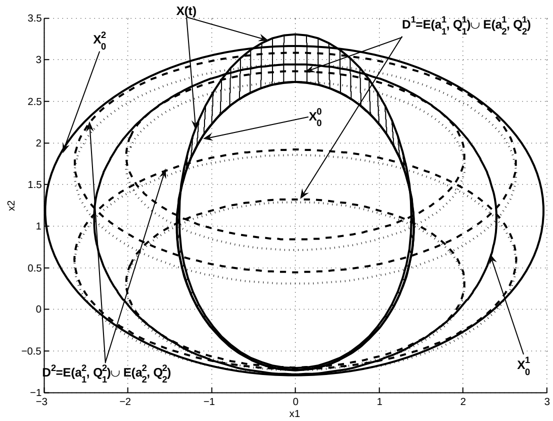

Figure 1: The reachable set $X(t)$ of system (7) and its external ellipsoidal estimates received at the first and second iteration

Figure 1 presents some iterations of the algorithm.

The first step of the first iteration is shown at Figure 1 by dotted lines. At the second step of the first iteration we get two new ellipsoids $E(a_1^1, Q_1^1)$, $E(a_2^1, Q_2^1)$ indicated by chain lines. After that we get ellipsoid X_0^1 containing the union $D^1 = E(a_1^1, Q_1^1) \bigcup E(a_2^1, Q_2^1)$ and so we finish the first iteration. Similarly we get X_0^2 during second iteration.

So the reachable set $X(t_i; t_0, X_0)$ of system (7)–(8) is covered (with enough high accuracy) by the union of two ellipsoids $D^i = E(a_1^i, Q_1^i) \bigcup E(a_2^i, Q_2^i)$:

$$X(t_i; t_0, X_0) \subseteq D^i + o(h)B(0,1), \ t_i = t_0 + ih \leq T.$$

Remark. Comparing this non–convex set-valued estimate with the convex estimate by one ellipsoidal tube [2] we find that this new non–convex estimate of reachable set may be more precise. Nevertheless both approaches may be used simultaneously to get better estimation results.

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