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UNITARY EQUIVALENCE TO INTEGRAL OPERATORS AND AN APPLICATION

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Abstract: In this paper, we characterize all closed, linear, densely defined operators in a separable Hilbert space which are unitarily equivalent to a Carleman integral operator in $L^2(\mathbb{R})$ whose kernel and Carleman functions are infinitely smooth and vanish at infinity together with all derivatives. There is an application to third-kind integral equations.

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1. Preliminary Notions and Definitions

Throughout, $\mathcal{H}$ is a complex, separable, infinite dimensional Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$ and inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, and the symbols $\mathbb{C}$ and $\mathbb{N}$ refer to the complex plane and the set of all positive integers, respectively. Let $\mathfrak{C}(\mathcal{H})$ be the set of all closed, linear, densely defined operators in $\mathcal{H}$, and let $\mathfrak{R}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For an operator S in $\mathfrak{C}(\mathcal{H})$, D_S stands for a linear manifold that is the domain of S, and S^* for the adjoint to S (w.r.t. $\langle\cdot,\cdot\rangle_{\mathcal{H}}$). We let $\mathfrak{C}_0(\mathcal{H})$ denote the collection of all those operators S in $\mathfrak{C}(\mathcal{H})$ for which there exists an orthonormal sequence $\{e_n\} \subset D_{S^*}$ such that $\lim_{n\to\infty} \|S^*e_n\|_{\mathcal{H}} = 0$, and we let $\mathfrak{C}_{00}(\mathcal{H})$ denote the subset of $\mathfrak{C}_0(\mathcal{H})$ consisting of all those operators S in $\mathfrak{C}(\mathcal{H})$ for which there exist a dense linear manifold D in $\mathcal{H}$ and an orthonormal sequence $\{e_n\}$ such that $\{e_n\} \subset D \subset$

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$D_S \cap D_{S^*}$, $\lim_{n\to\infty} \|Se_n\|_{\mathcal{H}} = 0$, and $\lim_{n\to\infty} \|S^*e_n\|_{\mathcal{H}} = 0$. If $T \in \mathfrak{R}(\mathcal{H})$, define the operator set $\mathcal{M}(T) = (T\mathfrak{R}(\mathcal{H}) \cup T^*\mathfrak{R}(\mathcal{H})) \cap (\mathfrak{R}(\mathcal{H})T \cup \mathfrak{R}(\mathcal{H})T^*)$, where $S\mathfrak{R}(\mathcal{H}) = \{SV \mid V \in \mathfrak{R}(\mathcal{H})\}$, $\mathfrak{R}(\mathcal{H})S = \{VS \mid V \in \mathfrak{R}(\mathcal{H})\}$. A factorization of an operator $T \in \mathfrak{R}(\mathcal{H})$ into the product $T = WV^*$, provided that the operators V, $W \in \mathfrak{R}(\mathcal{H})$ are subject to the provisos $VV^* \in \mathcal{M}(T)$, $WW^* \in \mathcal{M}(T)$, will be called an *$\mathcal{M}$ factorization for T*.

Let $\mathbb{R}$ be the real line equipped with the Lebesgue measure, and let $L^2 = L^2(\mathbb{R})$ be the Hilbert space of (equivalence classes of) measurable complex-valued functions on $\mathbb{R}$ equipped with the inner product $\langle f, g\rangle_{L^2} = \int_{\mathbb{R}} f(s)\overline{g(s)}\,ds$ and the norm $\|f\|_{L^2} = \sqrt{\langle f, f\rangle_{L^2}}$. An operator $T \in \mathfrak{C}(L^2)$ is said to be *integral* if there exists a measurable function $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{C}$, a *kernel*, such that, for each $f \in D_T$, $(Tf)(s) = \int_{\mathbb{R}} \boldsymbol{T}(s,t)f(t)\,dt$ for almost every s in $\mathbb{R}$. A kernel $\boldsymbol{T}$ is said to be *Carleman* if $\boldsymbol{T}(s,\cdot) \in L^2$ for almost every fixed s in $\mathbb{R}$. To each Carleman kernel $\boldsymbol{T}$ there corresponds a *Carleman function* $\boldsymbol{t} : \mathbb{R} \to L^2$ defined by $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s,\cdot)}$ for all s in $\mathbb{R}$ for which $\boldsymbol{T}(s,\cdot) \in L^2$. The Carleman kernel $\boldsymbol{T}$ is called *bi-Carleman* in case its conjugate transpose kernel $\boldsymbol{T}_*$ ($\boldsymbol{T}_*(s,t) = \overline{\boldsymbol{T}(t,s)}$) is also a Carleman kernel.

Throughout, $C(X,B)$, where B is a Banach space (with norm $\|\cdot\|_B$), denotes the Banach space (with the norm $\|f\|_{C(X,B)} = \sup_{x\in X} \|f(x)\|_B$) of continuous B-valued functions defined on a locally compact space X and vanishing at infinity. We say that the series $\sum_n f_n$ *au-converges in* $C(X,B)$ if $f_n \in C(X,B)$ ($n \in \mathbb{N}$) and $\sup_{x\in X} \sum_{n=l}^{m} \|f_n(x)\|_B \to 0$ as l, $m \to \infty$.

A bi-Carleman kernel $\boldsymbol{T}$ is called a K^∞ *kernel* [8] if: (i) the function $\boldsymbol{T}$ and all its partial derivatives of all orders on $\mathbb{R}^2$ are in $C\left(\mathbb{R}^2, \mathbb{C}\right)$, (ii) the Carleman function $\boldsymbol{t}$, $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s,\cdot)}$, and its (strong) derivatives $\boldsymbol{t}^{(i)}$ of all orders on $\mathbb{R}$ are in $C\left(\mathbb{R}, L^2\right)$, and (iii) the Carleman function $\boldsymbol{t}_*$, $\boldsymbol{t}_*(s) = \overline{\boldsymbol{T}_*(s,\cdot)} = \boldsymbol{T}(\cdot, s)$, and its (strong) derivatives $\boldsymbol{t}_*^{(j)}$ of all orders on $\mathbb{R}$ are in $C\left(\mathbb{R}, L^2\right)$. A Carleman kernel $\boldsymbol{T}$, which satisfies only the two above conditions (i), (ii), is called an SK^∞ *kernel* [8]. We also say that the K^∞ kernel $\boldsymbol{T}$ of an integral operator $T \in \mathfrak{R}(L^2)$ is of *Mercer type* if every operator belonging to $\mathcal{M}(T)$ is also an integral operator with a K^∞ kernel.

Recall also that a bounded linear operator $U : \mathcal{H} \to L_2$ is *unitary* if U has range L^2 and $\langle Uf, Ug\rangle_{L^2} = \langle f, g\rangle_{\mathcal{H}}$ for all f, $g \in \mathcal{H}$.

2. Results on Unitary Equivalences

Theorem 1. (see [8], [6]) *An operator S belongs to the class $\mathfrak{C}_0(\mathcal{H})$ (resp.,*

$\mathfrak{C}_{00}(\mathcal{H})$) if and only if there exists a unitary operator $U : \mathcal{H} \to L^2$ such that the operator $T = USU^{-1}$ is an integral operator with an SK^∞ (resp., K^∞) kernel.

Theorem 2. (see [7]) *Suppose that for a family $\{S_r \mid r \in \mathbb{N}\} \subset \mathfrak{R}(\mathcal{H})$ there exists an orthonormal sequence $\{e_n\} \subset \mathcal{H}$ such that $\sup_{r\in\mathbb{N}} \|S_r^* e_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. Then there exists a unitary operator $U : \mathcal{H} \to L_2$ such that for each $r \in \mathbb{N}$ the operator $T_r = US_rU^{-1}$ is an integral operator with an SK^∞ kernel.*

Theorem 3. (see [5]) *Suppose that for a family $\{S_\alpha \mid \alpha \in \mathcal{A}\} \subset \mathfrak{R}(\mathcal{H})$ there exists an orthonormal sequence $\{e_n\} \subset \mathcal{H}$ such that $\sup_{\alpha\in\mathcal{A}} \|S_\alpha^* e_n\|_{\mathcal{H}} \to 0$, $\sup_{\alpha\in\mathcal{A}} \|S_\alpha e_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. Then there exists a unitary operator $U : \mathcal{H} \to L^2$ such that all the operators $T_\alpha = US_\alpha U^{-1}$ ($\alpha \in \mathcal{A}$) and their linear combinations are integral operators with K^∞ kernels of Mercer type.*

It should be stressed that in the proofs of the theorems just stated the action of each unitary $U : \mathcal{H} \to L^2$ is determined explicitly, by specifying two concrete orthonormal bases, of $\mathcal{H}$ and of L^2, one of which is meant to be the image by U of the other; the basis for L_2 may be chosen to be an infinitely smooth wavelet basis.

Theorem 4. *Suppose that $T \in \mathfrak{R}(L^2)$ is an integral operator such that its kernel $\boldsymbol{T}$ is a K^∞ kernel of Mercer type. Then, for any $\mathcal{M}$ factorization $T = WV^*$ of T and for any orthonormal basis $\{u_n\}$ for L^2, the following formulae hold*

$$\frac{\partial^{i+j}\boldsymbol{T}}{\partial s^i \partial t^j}(s,t) = \sum_n (Wu_n)^{(i)}(s)\overline{(Vu_n)^{(j)}(t)}, \tag{1}$$

$$\boldsymbol{t}^{(i)}(s) = \sum_n \overline{(Wu_n)^{(i)}(s)}Vu_n, \quad \boldsymbol{t}_*^{(j)}(t) = \sum_n \overline{(Vu_n)^{(j)}(t)}Wu_n \tag{2}$$

$$(Tf)^{(i)}(s) = \sum_n \langle f, Vu_n\rangle_{L^2} (Wu_n)^{(i)}(s), \tag{3}$$

for all non-negative integers i, j, all s, $t \in \mathbb{R}$, and all $f \in L^2$, where the series of (1) au-converges in $C(\mathbb{R}^2, \mathbb{C})$, the series of (2) converge in $C(\mathbb{R}, L^2)$, and the series of (3) au-converges in $C(\mathbb{R}, \mathbb{C})$.

On the one hand, the result presents universal formulae for recapturing a Mercer-type K^∞ kernel together with all its derivatives from its associated operator by $\mathcal{M}$ factoring the latter. On the other hand, the result generalizes both Mercer's Theorem [10] (about uniform convergence of bilinear eigenfunction expansions for positive definite, continuous kernels on compact domains) and Kadota's Theorem [1, 2] (about termwise differentiation of those expansions) to various other settings, including the unboundedness of the domain of kernels, and the non-compactness and the non-selfadjointness of the induced integral operators.

3. An Application to Third-Kind Integral Equations

Let (X, μ) be a σ-finite non-atomic separable measure space. The *general integral equation of the third kind* in $L^2(X, \mu)$ is an equation of the form

$$\boldsymbol{A}(s)x(s) - \lambda \int_X \boldsymbol{T}(s,t)x(t)d\mu(t) = y(s) \quad \text{a.e,} \tag{4}$$

where $\boldsymbol{A}$ (the coefficient of the equation) is a given bounded μ-measurable function, $\boldsymbol{T}$ (the kernel of the equation) is a given $\mu \times \mu$-measurable function inducing the bounded integral operators $(Tx)(s) = \int_X \boldsymbol{T}(s,t)x(t)\,d\mu(t)$, $(T^*x)(s) = \int_X \overline{\boldsymbol{T}(t,s)}x(t)\,d\mu(t)$ on $L^2(X, \mu)$, the scalar $\lambda \in \mathbb{C}$ is given, $y \in L^2(X, \mu)$ is given, and $x \in L^2(X, \mu)$ is to be determined.

Theorem 5. *If $\alpha \in \mathbb{C}$ is such that $\mu\{s \in X \mid |\boldsymbol{A}(s) - \alpha| < \varepsilon\} > 0$ for all $\varepsilon > 0$, then the equation* (4) *is (unitarily) equivalent to the integral equation in L^2*

$$\alpha f(s) + \int_{\mathbb{R}} \boldsymbol{K}_\lambda(s,t)f(t)dt = g(s) \quad \text{a.e.}, \tag{5}$$

where the function $g \in L_2$ is given, the function $f \in L_2$ is to be determined, and, for each $\lambda \in \mathbb{C}$, $\boldsymbol{K}_\lambda$ is a K^∞ kernel of Mercer type having the form $\boldsymbol{K}_\lambda(s,t) = \boldsymbol{H}(s,t) - \lambda\boldsymbol{G}(s,t)$, where both K^∞ kernels $\boldsymbol{H}$ and $\boldsymbol{G}$ are independent of λ.

The proof relies on the observation that if A is the multiplication operator induced on $L^2(X, \mu)$ by $\boldsymbol{A}$, and I is identity operator on $L^2(X, \mu)$, then the family $\{S_1 = A - \alpha I, S_2 = T, S_3 = (|S_1|^2 + |T|^2)^{1/2}, S_4 = (|S_1^*|^2 + |T^*|^2)^{1/2}\}$ (where $|S| = (S^*S)^{1/2}$) satisfies conditions of Theorem 3; the corresponding sequence $\{e_n\}$ can be formed by the Rademacher functions in $L^2(X, \mu)$ (see, e.g., [3]). Therefore there is a unitary operator $U : L^2(X, \mu) \to L^2$ such that the operators $T_i = US_iU^{-1}$ $(i = 1, 2, 3, 4)$ and their linear combinations are integral operators with K^∞ kernels of Mercer type. Then one can write (see (4)) $Uy = U(A - \lambda T)U^{-1}Ux = U(\alpha I + S_1 - \lambda S_2)U^{-1}Ux = \alpha f + (T_1 - \lambda T_2)f = g$ with $f = Ux$, $g = Uy$, so the kernels $\boldsymbol{H}$ and $\boldsymbol{G}$ in the theorem are just those K^∞ kernels of Mercer type that induce T_1 and T_2, respectively. The K^∞ kernels of T_3 and of T_4 will be denoted by $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$, respectively.

In at least one special case, namely when $\widetilde{\gamma} = \int_{\mathbb{R}} \widetilde{\boldsymbol{F}}(s,s)\,ds < \infty$, $\gamma =$

$\int_{\mathbb{R}} \boldsymbol{F}(s,s)\,ds < \infty$ and $\alpha = 1$, the solution of the equation (5) is transparent, and is as follows. For each n, $p = 0, 1, 2 \dots$ and s_i, $t_i \in \mathbb{R}$ $(1 \le i \le p)$, define a polynomial in λ of degree $\le p + n$ by

$$B_{n,p}(\boldsymbol{K}_\lambda) := \frac{1}{n!} \int\limits_{\mathbb{R}} \dots \int\limits_{\mathbb{R}} \boldsymbol{K}_\lambda \begin{pmatrix} s_1 & \dots & s_p & \xi_1 & \dots & \xi_n \\ t_1 & \dots & t_p & \xi_1 & \dots & \xi_n \end{pmatrix} d\xi_1 \dots d\xi_n \,,$$

where $\boldsymbol{K}_\lambda \begin{pmatrix} x_1 & \dots & x_\nu \\ y_1 & \dots & y_\nu \end{pmatrix} = \det\left[\boldsymbol{K}_\lambda(x_i, y_j)\right]_{i,j=1}^{\nu}$ if $\nu > 0$ and $= 1$ if $\nu = 0$. Then it can be proved by using Theorem 4 that, for each λ of any fixed compact set $\mathfrak{K}$ in $\mathbb{C}$,

$$|B_{n,0}(\boldsymbol{K}_\lambda)| \le C_{n,\mathfrak{K}}, \quad \|B_{n,p}(\boldsymbol{K}_\lambda)\|_{C(\mathbb{R}^{2p},\mathbb{C})} \le M^{2p} c_{\mathfrak{K}}^p C_{n,\mathfrak{K}},$$
$$\|B_{n,p}(\boldsymbol{K}_\lambda)\|_{C(\mathbb{R}^{2p-1},L^2)} \le (\max\{\widetilde{\gamma}, \gamma\})^{\frac{1}{2}} M^{2p-1} c_{\mathfrak{K}}^p C_{n,\mathfrak{K}},$$
$$\|B_{n,p}(\boldsymbol{K}_\lambda)\|_{C(\mathbb{R}^{2p-2},L^2(\mathbb{R}^2))} \le (\widetilde{\gamma}\gamma)^{\frac{1}{2}} M^{2p-2} c_{\mathfrak{K}}^p C_{n,\mathfrak{K}},$$

with $M^2 = \max\left\{\sup\limits_{s\in\mathbb{R}} \widetilde{\boldsymbol{F}}(s,s), \sup\limits_{t\in\mathbb{R}} \boldsymbol{F}(t,t)\right\}$, $c_{\mathfrak{K}}^2 = 2\sup\limits_{\mu\in\mathfrak{K}}(1 + |\mu|^2)$, and $C_{n,\mathfrak{K}} = c_{\mathfrak{K}}^n \left(\widetilde{\gamma}^n + \gamma^n\right)/2n!$. Thus, we have the following:

Theorem 6. *The polynomial pth series* $\sum_{n=0}^{\infty} B_{n,p}(\boldsymbol{K}_\lambda)$ $(p \ge 0)$ *of Fredholm type for* $\boldsymbol{K}_\lambda$ *is absolutely convergent in* $\mathbb{C}$ $(p = 0)$, *in* $C\left(\mathbb{R}^{2p}, \mathbb{C}\right)$ $(p \ge 1)$, *in* $C\left(\mathbb{R}^{2p-1}, L^2\right)$ $(p \ge 1)$, *and in* $C\left(\mathbb{R}^{2p-2}, L^2\left(\mathbb{R}^2\right)\right)$ $(p > 1)$, *uniformly in* λ *on every compact subset in* $\mathbb{C}$.

Let $\Delta_p^{(j)}\left[\boldsymbol{K}_\lambda\right]$ denote the jth derivative w.r.t. λ of an entire function of λ that is the sum of the Fredholm-type polynomial pth series $\sum_{n=0}^{\infty} B_{n,p}(\boldsymbol{K}_\lambda)$, viewed as taking values in $C\left(\mathbb{R}^{2p}, \mathbb{C}\right)$ if $p \ge 1$, and in $\mathbb{C}$ if $p = 0$. Let θ_p denote the zero in the space $C\left(\mathbb{R}^{2p}, \mathbb{C}\right)$, and let $\theta_0 = 0$.

The next theorem strongly resembles the Fredholm solution of the Fredholm integral equation of the second kind for a given value of the spectral parameter [9] (for an extended version of the theorem below, see also [4]).

Theorem 7. *Let* λ_0 *be any fixed point in* $\mathbb{C}$. *Then: (A) There are unique non-negative integers* $\mathbf{d} = \mathbf{d}(\lambda_0)$ *and* $\mathbf{r} = \mathbf{r}(\lambda_0)$ *such that* $\Delta_{\mathbf{d}}^{(\mathbf{r})}(\boldsymbol{K}_{\lambda_0}) \ne \theta_{\mathbf{d}}$, $\Delta_p^{(k)}(\boldsymbol{K}_{\lambda_0}) = \theta_p$ *if* $0 \le k < \mathbf{r}$, $p \ge 0$, *nd* $\Delta_p^{(\mathbf{r})}(\boldsymbol{K}_{\lambda_0}) = \theta_p$ *if* $0 \le p < \mathbf{d}$; *(B) If the point* $\begin{pmatrix} s_1' & \dots & s_{\mathbf{d}}' \\ t_1' & \dots & t_{\mathbf{d}}' \end{pmatrix} \in \mathbb{R}^{2\mathbf{d}}$ *satisfies* $\Delta_{\mathbf{d}}^{(\mathbf{r})}\left[\boldsymbol{K}_{\lambda_0}\right] \begin{pmatrix} s_1' & \dots & s_{\mathbf{d}}' \\ t_1' & \dots & t_{\mathbf{d}}' \end{pmatrix} = \delta \ne 0$, *then the functions* $\phi_i(s) = \Delta_{\mathbf{d}}^{(\mathbf{r})}\left[\boldsymbol{K}_{\lambda_0}\right] \begin{pmatrix} s_1' & \dots & s_{i-1}' & s & s_{i+1}' & \dots & s_{\mathbf{d}}' \\ t_1' & \dots & \dots & \dots & \dots & \dots & t_{\mathbf{d}}' \end{pmatrix}$ $(s \in \mathbb{R}, 1 \le i \le \mathbf{d})$ *form a base for the set of solutions* f *of the homogeneous equation* $f(s) + \int_{\mathbb{R}} \boldsymbol{K}_{\lambda_0}(s,t) f(t)\,dt = 0$, *and the functions* $\psi_l(t) =$

$\overline{\Delta_{\mathbf{d}}^{(\mathbf{r})}\left[\boldsymbol{K}_{\lambda_0}\right]\begin{pmatrix} s'_1 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & s'_{\mathbf{d}} \\ t'_1 & \ldots & t'_{l-1} & t & t'_{l+1} & \ldots & t'_{\mathbf{d}} \end{pmatrix}}$ $(t \in \mathbb{R}, 1 \leq l \leq \mathbf{d})$ *form a base for the set of solutions f of the conjugate homogeneous equation*

$$f(t) + \int_{\mathbb{R}} \overline{\boldsymbol{K}_{\lambda_0}(s,t)} f(s)\, ds = 0\,,$$

(C) The equation $f(s) + \int_{\mathbb{R}} \boldsymbol{K}_{\lambda_0}(s,t) f(t)\, dt = g(s)$ is soluble if and only if $\langle g, \psi_l \rangle_{L^2} = 0$ $(1 \leq l \leq \mathbf{d})$ and the general solution is then given by $f(s) = g(s) - \frac{1}{\delta} \int_{\mathbb{R}} \Delta_{\mathbf{d}+1}^{(\mathbf{r})}\left[\boldsymbol{K}_{\lambda_0}\right]\begin{pmatrix} s & s'_1 & \ldots & s'_{\mathbf{d}} \\ t & t'_1 & \ldots & t'_{\mathbf{d}} \end{pmatrix} g(t)\, dt + \sum_{i=1}^{\mathbf{d}} c_i \phi_i(s)$, where $c_1, \ldots, c_{\mathbf{d}}$ are arbitrary complex constants.

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