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QUASI-NORMAL SCALE ELIMINATION THEORY OF TURBULENCE

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Abstract: We present an analytical theory of turbulence based upon the procedure of successive elimination of small-scale modes that leads to gradual coarsening of the flow field and accumulation of eddy viscosity. The Reynolds number based upon the eddy viscosity remains $O(1)$. The main results of the theory are analytical expressions for eddy viscosity and kinetic energy spectrum. Partial scale elimination yields a subgrid-scale representation for large eddy simulations while the elimination of all fluctuating scales is analogous to the Reynolds averaging.

Key Words: analytical turbulence theories, eddy viscosity

1. Introduction

Fluid turbulence is a complicated, strongly nonlinear, multiscale phenomenon which presents a fundamental unresolved challenge. In this paper, we elaborate a new analytical theory of turbulence which is maximally proximate to first principles. This theory is based upon successive elimination of fluctuating small-

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scale modes and calculation of compensating corrections to the viscosity. The theory formulation as well as the computation of the eddy viscosity and kinetic energy spectrum are presented in the next section. Section 3 is discussion and conclusions.

2. The QNSE Model and Main Results

We consider homogeneous, isotropic, three-dimensional turbulent flows described by the Navier–Stokes and continuity equations,

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_i^0 - \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu_0 \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2}$$

where u_i is the fluctuating velocity, P is the pressure, ρ is the constant density, ν_0 is the molecular viscosity. Equation (1) is strongly nonlinear. Large-scale instabilities excite secondary flows that form structures populating smaller scales. Nonlinear interactions between the structures on different scales generate highly irregular, stochastic flow field known as turbulence. The force, $f_i^0(\mathbf{x}, t)$, is an explicit external forcing that maintains turbulence in a statistically steady state. This large-scale energy injection generates Kolmogorov cascade down to the smallest scales. The dynamics of this cascade is independent of the details of the force.

A space-time Fourier transform of the three-dimensional (3D) velocity field, $u_i(\mathbf{x}, t)$, in the domain bounded by the wave number of the viscous dissipation, $k_d = O[(\epsilon/\nu_0^3)^{1/4}]$, ϵ being the rate of the viscous dissipation, is

$$u_i(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int_{k \le k_d} \mathrm{d}\mathbf{k} \int \mathrm{d}\omega \; u_i(\mathbf{k}, \omega) \exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)]. \tag{3}$$

Fourier-transformed continuity equation is

$$k_i u_i(\hat{k}) = 0. \tag{4}$$

Using this equation, we exclude pressure in the Fourier-transformed equation (1) and present it as

$$u_l(\hat{k}) = G(\hat{k}) f_l^0(\hat{k}) - \frac{\mathrm{i}}{2} G(\hat{k}) P_{lmn}(\mathbf{k}) \int u_m(\hat{q}) u_n(\hat{k} - \hat{q}) \frac{\mathrm{d}\hat{q}}{(2\pi)^4}, \tag{5}$$

where $\hat{k} = (\mathbf{k}, \omega)$, $\hat{q} = (\mathbf{q}, \Omega)$, and the Einstein summation rule is enforced. The projector operators,

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2, \tag{6}$$

$$P_{lmn}(\mathbf{k}) = k_m P_{ln}(\mathbf{k}) + k_n P_{lm}(\mathbf{k}), \tag{7}$$

are the result of the incompressibility of the flow field;

$$G^{-1}(\hat{k}) = G^{-1}(\mathbf{k}, \omega) = -i\omega + \nu_0 k^2 \tag{8}$$

is the Green function, and δ_{ij} is the Kronecker δ-symbol.

The system (4), (5) is defined in the entire domain $k \in (0, k_d]$. For an arbitrary wave number $\Lambda \leq k_d$ and small shell $\Delta\Lambda$, we define the domains $\mathcal{D}^< = (0, \Lambda - \Delta\Lambda]$ and $\mathcal{D}^> = (\Lambda - \Delta\Lambda, \Lambda]$ such that $\mathcal{D} = (0, \Lambda] = \mathcal{D}^< \cup \mathcal{D}^>$. We now define *slow* modes, $u_l^<(\mathbf{k}, \omega)$, and *fast* modes, $u_l^>(\mathbf{k}, \omega)$, according to

$$u_l(\mathbf{k}, \omega) = \begin{cases} u_l^<(\mathbf{k}, \omega) & \text{for } k \in \mathcal{D}^< \\ u_l^>(\mathbf{k}, \omega) & \text{for } k \in \mathcal{D}^>. \end{cases} \tag{9}$$

The coarse-grained description of the system is achieved by a systematic procedure of successive ensemble averaging over the fast modes which we refer to as the "small scale elimination." This procedure coarsens the 'grain' of the spectral representation of the flow domain by the measure of $\mathcal{D}^>$. Simultaneously, the viscosity is modified after each iteration. The process starts at $\Lambda = k_d$ and continues to arbitrarily small Λ. The ensuing coarse-grained equation preserves the analytical structure of the Navier–Stokes equation but its effective, or eddy viscosity becomes Λ-dependent. Accordingly, the expression for the Green function, equation (8), will contain the eddy viscosity rather than ν_0.

Using (9), vector u_l can be decomposed into a sum, $u_l = u_l^< + u_l^>$. The nonlinearity in the Navier-Stokes equation (5) couples the fast and slow modes. Using (5), we can write expressions for either $u_l^<(\mathbf{k}, \omega)$ or $u_l^>(\mathbf{k}, \omega)$,

$$\begin{aligned} u_l(\hat{k}) &= G(\hat{k}) f_l^0(\hat{k}) - \frac{\mathrm{i}}{2} G(\hat{k}) P_{lmn}(\mathbf{k}) \int [u_m^<(\hat{q}) u_n^<(\hat{k} - \hat{q}) \\ &+ 2u_m^>(\hat{q}) u_n^<(\hat{k} - \hat{q}) + u_m^>(\hat{q}) u_n^>(\hat{k} - \hat{q})] \frac{\mathrm{d}\hat{q}}{(2\pi)^4}. \end{aligned} \tag{10}$$

Further derivations are conveniently performed using the method of Feynman diagrams. We introduce the following notations:

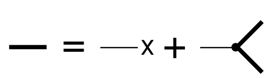
$u_l(\hat{k})$ =

$G(\hat{k})$ =

$f_l^0(\hat{k})$ = $\times$

$-\frac{i}{2} P_{lmn}(\mathbf{k}) \int \frac{d\hat{q}}{(2\pi)^{d+1}}$ = $\bullet$

The Navier-Stokes equation (5) can be represented by the following diagrammatic equation:



The thick line denotes the slow velocity modes,

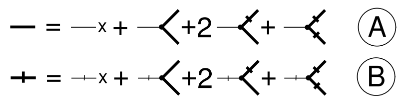
$u_i^<(\hat{k})$ =

while a thick crossed line represents the fast velocity modes,

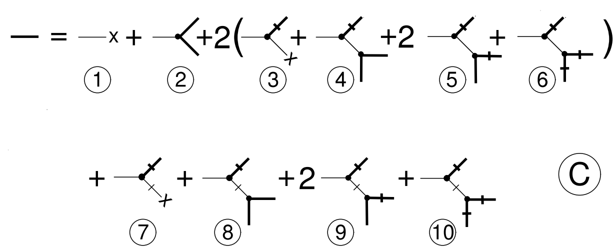
$u_i^>(\hat{k})$ = 

Generally, all uncrossed lines will denote slow variables for which $k \in \mathcal{D}^<$, and all crossed lines will refer to fast variables for which $k \in \mathcal{D}^>$.

Using (10), the equations for slow and fast modes can be represented by the following diagrams:



Utilizing equations A and B, we can derive a perturbative equation for $u_i^<(\hat{k})$. The third term on the right side of A is the cross-term which is evaluated by substituting equation A in itself. The last term on the right side of A is computed by substituting equation B. The resulting diagrammatic equation for $u_i^<(\hat{k})$ takes the form



To exclude the fast modes from equation C, one needs to perform ensemble averaging. This procedure utilizes statistical properties of the fast modes.

We can rearrange equation (10) in the form of the Langevin equation,

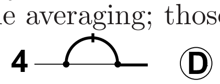
$$u_l(\hat{k}) = G(\hat{k}) f_l(\hat{k}), \tag{11}$$

where $f_l(\hat{k})$ is the *modal* force that represents random agitation exerted by the modes upon each other via nonlinear interactions. This force is solenoidal, zero-mean, white noise in time, isotropic and homogeneous in time and space, such that its correlation function is

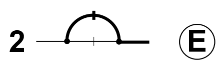
$$\langle f_i(\mathbf{k}, \omega) f_j(\mathbf{k}', \omega') \rangle = 2D(2\pi)^4 k^{-3} P_{ij}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega'). \tag{12}$$

The correlator (12) accounts for the mean energy input to a mode k due to its interaction with other modes. Thus, its amplitude, D, is proportional to the mean rate of energy transfer through the mode k, i.e., $D = C\epsilon$. Yakhot and Orszag [6] have computed the coefficient in this relationship, $C \simeq 15.7$. The factor k^{-3} in (12) ensures correct dimensionality. Several constant coefficients are introduced for convenience.

Upon replacement of all fast velocity modes in equation C by the Langevin equation (11) one can perform ensemble averaging of each term. Diagrams 4 and 8 yield zeroes as the first moments of the fluctuating force $\mathbf{f}^{>}$. Diagrams 6 and 10 become triple moments of the fluctuating force and they are assumed to be zero. Obviously, a Gaussian force would have fulfilled this condition. The requirement of zero triple moments in the shell $\mathcal{D}^{>}$ is, however, much weaker than the Gaussianity. Loosely, we shall refer to the force $\mathbf{f}^{>}$ as quasi-normal and note that this assumption places the present theory in the class of the quasi-normal closures (Orszag [2], McComb [1]). Furthermore, the diagram 3 represents a convolution of the kind $\int u^{>}(\hat{k}-\hat{q})G(\hat{q})f^{0}(\hat{q})\mathrm{d}\mathbf{q} \propto G(\hat{k})u^{>}(\hat{k}), k \in \mathcal{D}^{<}$ by the delta-function-like nature of the external force. But $u^{>}(\hat{k}) = 0$ for $k \in \mathcal{D}^{<}$ by definition of $u^{>}$. The diagram 7 is also zero as representing an infinitesimal correction to $f_j^0(\mathbf{k},\omega)$. Thus, only the diagrams 5 and 9 give non-zero contributions after the averaging; those are



and



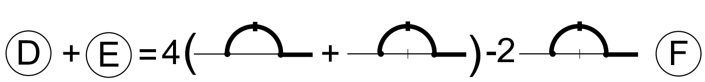
The sum of these diagrams yields correction to the inverse Green function which in turn generates corrections to the eddy viscosity due to the coarse-graining. In these diagrams, the thick half-circle denotes the spectrum tensor $U_{m\mu}(\hat{q}),\ q \in \mathcal{D}^{>}$, defined via the two-point, two-time velocity correlation function,

$$\langle u_m(\mathbf{q},\Omega)u_\mu(\mathbf{q}',\Omega')\rangle = (2\pi)^4\delta(\mathbf{q}+\mathbf{q}')\delta(\Omega+\Omega')U_{m\mu}(\mathbf{q},\Omega). \tag{13}$$

Using equations (11), (12), and (13), we evaluate the spectrum tensor,

$$U_{m\mu}(\hat{q}) = 2Dq^{-3}|G(\hat{q})|^2 P_{m\mu}(\mathbf{q}). \tag{14}$$

Next, we represent the sum of the diagrams D and E as



and take a note that the analytical expressions for both diagrams in the brackets are identical,

$$-\frac{1}{4}G(\hat{k})P_{lmn}(\mathbf{k})\int U_{m\mu}(\hat{q})G(\hat{k}-\hat{q})P_{n\mu\sigma}(\mathbf{k}-\mathbf{q})u_\sigma^{<}(\hat{k})\frac{\mathrm{d}\hat{q}}{(2\pi)^4}, \tag{15}$$

albeit their integration domains are different. The first integral is computed over the intersection of $q \in \mathcal{D}^>$ and $|\mathbf{k}-\mathbf{q}| \in \mathcal{D}^<$ while the second integral is computed over the intersection of $q \in \mathcal{D}^>$ and $|\mathbf{k}-\mathbf{q}| \in \mathcal{D}^>$. These domains do not intersect so that the sum of the two integrals can be represented by a single integral over the union of their domains. This union is the intersection of $q \in \mathcal{D}^>$ and $|\mathbf{k}-\mathbf{q}| \in \mathcal{D}^< \cup \mathcal{D}^>$ equal to the intersection $\mathcal{D}^> \cap D$ or, simply, the shell $q \in \mathcal{D}^>$ itself. In the remaining integral in F the integration domain is the intersection of two spherical shells $\mathcal{D}^>$ shifted by k relative to each other. This domain is $O[(\Delta\Lambda)^2]$ as the intersection of two $O(\Delta\Lambda)$ shells. Eventually, a limit $\Delta\Lambda \to 0$ will be taken such that the $O[(\Delta\Lambda)^2]$ terms should be neglected.

Finally, equation (15) can be re-arranged as

$$-\frac{C\epsilon G(\hat{k})}{2(2\pi)^4} P_{lmn}(\mathbf{k}) u_\sigma^<(\hat{k}) \int^> |G(\hat{q})|^2 G(\hat{k}-\hat{q}) P_{n\mu\sigma}(\mathbf{k}-\mathbf{q}) P_{\mu m}(\mathbf{q}) q^{-3} \mathrm{d}\hat{q}\,, \tag{16}$$

where

$$\int^> \frac{\mathrm{d}\hat{q}}{(2\pi)^4} = \int_{\mathcal{D}^>} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} = \int_{\Lambda}^{\Lambda-\Delta\Lambda} \frac{\mathrm{d}q}{2\pi} \oint_{\partial S_2} \frac{\Sigma}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi}. \tag{17}$$

The integrals are calculated using the distant interaction approximation in which $k \ll q$ and only the terms up to $O(k^2)$ are retained. The details of the integration and calculation of the correction to the viscosity, $\Delta\nu$, are given in Sukoriansky et al [4]. The final result is

$$\Delta\nu = -\frac{1}{10\pi^2} \frac{C\epsilon}{\nu^2 \Lambda^5} \Delta\Lambda. \tag{18}$$

Taking the limit $\Delta\Lambda \to 0$, one obtains an ordinary differential equation whose solution is

$$\nu(\Lambda) = \left(\frac{3C\epsilon}{40\pi^2}\right)^{1/3} \Lambda^{-4/3} \simeq 0.5\epsilon^{1/3}\Lambda^{-4/3}, \quad \Lambda \ll k_d. \tag{19}$$

Knowing ν, we can compute the Green function, G, the velocity correlator, U, and the kinetic energy spectrum,

$$E(k) = (2\pi)^{-3} k^2 \int_{-\infty}^{\infty} U_{\alpha\alpha}(k,\omega) d\omega = C_K \epsilon^{2/3} k^{-5/3}, \tag{20}$$

where $C_K = 1.62$ is the Kolmogorov constant.

Note that in the process of small scale elimination, the Reynolds number based upon the smallest resolvable scales remains $O(1)$.

3. Discussion and Conclusions

This paper presents the foundation of the new analytical theory of turbulence based upon successive small-scale modes elimination in the assumption that these modes abide quasi-normal statistics. The assumption of the quasi-normality is, in fact, the only major assumption of the QNSE theory. Another assumption, the distant interaction approximation, can be elaborated analytically and is a subject of the following up paper. Partial scale elimination yields a subgrid-scale eddy viscosity that can be utilized in large-eddy simulations. Complete scale elimination yields the eddy viscosity for RANS (Reynolds-averaged Navier-Stokes) models. Being simpler than the spectral models based upon the energy equation (e.g., Orszag [2], McComb [1]), the QNSE model can be extended to anisotropic turbulent flows with waves. For instance, the application of the QNSE theory to flows with stable stratification yields RANS models that are being utilized in models used for numerical weather prediction (Sukoriansky et al [5], [3]).

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