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THE STOKES EQUATIONS IN EXTERIOR n-DIMENSIONAL DOMAINS

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Abstract: The aim of this paper is the construction and representation of solutions u, p to the homogeneous Stokes equations

$$-\Delta u + \nabla p = 0 \quad \text{in} \quad G_e, \quad \nabla \cdot u = 0 \quad \text{in} \quad G_e, \quad u = \Phi \quad \text{on} \quad \Gamma, \qquad (S)$$

with methods of hydrodynamical potential theory. Here $G_e \subset R^n$ $(n \geq 2)$ is an exterior domain with boundary $\Gamma = \partial G_e \in C^2$, and $\Phi \in C^0(\Gamma)$ is some prescribed boundary value.

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1. Hydrodynamical Potential Theory

As in the classical potential theory we require Green formulas as a starting point, the hydrodynamical version of which has the following form. For sufficiently smooth, solenoidal vector functions u, v and scalar functions p, q in a bounded domain $A \subset \mathbb{R}^n$ $(n \geq 2)$ with the boundary $\partial A \in C^1$ we have Green's first and second formula

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$$\int\limits_A \Big(S\begin{matrix}u\\p\end{matrix}\Big)\cdot\binom{v}{q}\,dy \;=\; \int\limits_{\partial A}(T\begin{matrix}u\\p\end{matrix}N)\cdot v\,do \;+\; 2\int\limits_A Du\!:\!Dv\,dy, \tag{1}$$

$$\int\limits_A \Big\{\Big(S\begin{matrix}u\\p\end{matrix}\Big)\cdot\binom{v}{q} \;-\; \binom{u}{p}\cdot\Big(S'\begin{matrix}v\\q\end{matrix}\Big)\Big\}\,dy \;=\; \int\limits_{\partial A}\Big\{(T\begin{matrix}u\\p\end{matrix}N)\cdot v \;-\; u\cdot(T'\begin{matrix}v\\q\end{matrix}N)\Big\}\,do. \tag{2}$$

Here

$$S\;:\;\binom{u}{p}\longrightarrow S\begin{matrix}u\\p\end{matrix}\;:=\;\begin{pmatrix}-\Delta u+\nabla p\\ \nabla\cdot u\end{pmatrix},\qquad S'\;:\;\binom{u}{p}\longrightarrow S'\begin{matrix}u\\p\end{matrix}\;:=\;\begin{pmatrix}-\Delta u-\nabla p\\ -\nabla\cdot u\end{pmatrix}$$

stand for the formal Stokes operator and its adjoint, respectively, and

$$T\;:\;\binom{u}{p}\longrightarrow T\begin{matrix}u\\p\end{matrix}\;:=-2Du+pI_n,\qquad T'\;:\;\binom{u}{p}\longrightarrow T'\begin{matrix}u\\p\end{matrix}\;:=\;-2Du-pI_n$$

denote the stress tensors adjoint to each other. Here the deformation tensor is defined by

$$Du\;:=\;\tfrac{1}{2}(\nabla u+(\nabla u)^T) \tag{3}$$

with $(\nabla u)^T$ as the matrix transposed to $\nabla u:=(\partial_i u_k)_{k,i=1,\ldots,n}$. Here and in the following, I_n is always the $n\times n$ identity matrix and $N=N(y)$ the exterior (with respect to the bounded domain A) unit surface normal vector in $y\in\partial A$. For vectors $a,b\in I\!R^n$ and $n\times n$ matrices $C=(C_{ij}),\;\; D=(D_{ij})$ we set

$$a\cdot b:=\sum_{i=1}^n a_i b_i \;\;\text{and}\;\; C:D:=\sum_{i,j=1}^n C_{ij}D_{ij}.$$

Now the following uniqueness statement for classical solutions u,p of the equations (S) (i.e. $u\in C^2(G_e)\,\cap\,C^1(\overline{G_e}),\;\; p\in C^1(G_e)\,\cap\,C^0(\overline{G_e})$) can be proved using Green's first formula (1).

Lemma 1. *Let $G_e\subset I\!R^n$ $(n\geq 2)$ be an exterior domain with boundary $\Gamma\in C^2$ and $\Phi\in C^0(\Gamma)$. Then there exists at most one classical solution u,p to the equations (S), which satisfies the decay conditions*

$$\nabla^k u(x) \;=\; \mathcal{O}(|x|^{2-n-k}),\quad k=0,1, \tag{4}$$

$$p(x) \;=\; \mathcal{O}(|x|^{1-n}), \tag{5}$$

as $|x|\to\infty$.

Let $a_\infty\in I\!R^2$ be given. Then the condition (4) in the case $n=2,\;k=0$ may be weakened by

$$u(x)\;-\;a_\infty\ln|x|\;=\;\mathcal{O}(1). \tag{6}$$

Apart from the Green formulas (1) and (2), the explicit form of the Stokes fundamental tensor $E=(E_{jk})_{j,k=1,\ldots,n+1}$ is of a great importance. The following representation is well-known.

$n \geq 2 \quad (j, k = 1, \ldots, n)$:

$$
\begin{aligned}
E_{jk}(x) &= \frac{1}{2\omega_n}\left\{\frac{x_j x_k}{|x|^n} + \delta_{jk}\left\{\begin{array}{ll} \ln\frac{1}{|x|} & (n=2) \\ \frac{|x|^{2-n}}{n-2} & (n \geq 3) \end{array}\right\}\right\}, \\
E_{n+1,k}(x) &= E_{k,n+1}(x) = \frac{x_k}{\omega_n |x|^n}, \\
E_{n+1,n+1}(x) &= \delta(x).
\end{aligned} \tag{7}
$$

Here and in the future: ω_n always stands for the surface area of the $(n-1)$ dimensional unit sphere in $I\!R^n$ $(n \geq 2)$.

Now let u, p denote a solution to $S\genfrac{}{}{0pt}{}{u}{p} = \binom{0}{0}$ in A. Insert for v, q the column vector $E_k(x - \cdot) := (E_{jk}(x - \cdot))_{j=1,\ldots,n+1}$ as a function of y in Green's second formula (2), one after another for each $k = 1, \ldots, n+1$. Then, by transposing, we obtain:

$$
\int\limits_{\partial A} E^{(c)}(x-y)\, T\genfrac{}{}{0pt}{}{u}{p} N(y)\, do_y - \int\limits_{\partial A} D(x,y)\, u(y)\, do_y = \left\{\begin{array}{cl} -\binom{u}{p}(x) & ,\quad x \in A, \\ 0 & ,\quad x \notin \overline{A}. \end{array}\right. \tag{8}
$$

Here the $(n+1) \times n$ matrix $E^{(c)}$ is obtained from E by cancelling the last column $(\sim c)$ (note that $E = E^T$), and the $(n+1) \times n$ matrix $D(x,y)$ is defined by (use $T'_y E(x-y) = -T_x E(x-y)$)

$$
D(x,y) \;=\; (-T_x E(x-y)\, N(y)) \;=\; \Big((-T_x E_k(x-y))_{ij} N_j(y)\Big)_{ki}. \tag{9}
$$

The representation given in (8) is a so-called direct representation of a solution u, p to $S\genfrac{}{}{0pt}{}{u}{p} = \binom{0}{0}$ by its boundary values u and $T\genfrac{}{}{0pt}{}{u}{p}N$. This kind of representation is not unique, of course, since it depends on the corresponding Green formula. Besides the direct representations there are indirect representations by hydrodynamical surface potentials with general vector-valued source densities. For such densities $\Psi \in C^0(\Gamma)$ we therefore define the hydrodynamical single layer potential

$$
(E_n \Psi)(x) \;:=\; \int\limits_{\Gamma} E^{(c)}(x-y)\, \Psi(y)\, do_y, \quad x \notin \Gamma, \tag{10}
$$

and the hydrodynamical double layer potential

$$
(D_n \Psi)(x) \;:=\; \int\limits_{\Gamma} D(x,y)\, \Psi(y)\, do_y, \quad x \notin \Gamma. \tag{11}
$$

Both potentials are smooth functions outside the surface Γ and there they satisfy the homogeneous Stokes equations, i.e. we have

$$
SE_n\Psi = 0, \quad SD_n\Psi = 0.
$$

In the following, we will supply that part of these potentials referring to the velocity field u with a dot. This leads to the n–componental column vectors

$$(E_n^{\bullet}\Psi)(x) \;=\; \int\limits_{\Gamma} E^{(r,c)}(x-y)\,\Psi(y)\,do_y, \quad x \notin \Gamma, \tag{12}$$

$$(D_n^{\bullet}\Psi)(x) \;=\; \int\limits_{\Gamma} D^{(r)}(x,y)\,\Psi(y)\,do_y, \qquad x \notin \Gamma. \tag{13}$$

Here the $n \times n$ matrices $E^{(r,c)}$ and $D^{(r)}$ are defined by cancelling the last row ($\sim r$) in $E^{(c)}$ and D, respectively.

Besides the single layer potential $E_n\Psi$, in the following we also need its normal stresses $T(E_n\Psi)N$, which are defined in a neighbourhood $U \subset I\!R^n$ of the surface Γ for all $x \in U\backslash\Gamma$ and $\Psi \in C^0(\Gamma)$ by the n–componental column vector

$$\begin{aligned}(H_n^{\bullet}\Psi)(x) \;&=\; \int\limits_{\Gamma} T_x\Big(E^{(c)}(x-y)\,\Psi(y)\Big)\,N(\tilde{x})\,do_y \\ &=: \int\limits_{\Gamma} H(x,y)\,\Psi(y)\,do_y, \quad x \notin \Gamma.\end{aligned} \tag{14}$$

Here $\tilde{x} \in \Gamma$ is the uniquely determined projection of $x \in U$ onto Γ, and for the $n \times n$ matrix H we find

$$H(x,y) = \Big(D^{(r)}(y,x)\Big)^T = D^{(r)}(y,x) \quad \text{on} \quad \Gamma \times \Gamma. \tag{15}$$

Next we need some statements regarding the behaviour of the potentials in the point $x \in I\!R^n\backslash\Gamma$, if x passes through the surface Γ. Therefore let

$$G_i := I\!R^n \backslash \overline{G_e}$$

always denote the bounded complement of G_e. Then the following analogue to the Gauss integral formula of the classical potential theory holds (see Ladyzhenskaya [2, p. 56] in the case $n = 3$).

Lemma 2. *For the double layer potential $D_n^{\bullet}b$ with some constant density $b \in I\!R^n$ $(n \geq 2)$ we have*

$$(D_n^{\bullet}b)(x) \;=\; \begin{cases} b, & x \in G_i, \\ \frac{1}{2}b, & x \in \Gamma, \\ 0, & x \in G_e. \end{cases} \tag{16}$$

Let us now assume $z \in \Gamma$. With the definitions

$$\begin{aligned} w^i(z) \;&:=\; \lim_{\substack{x\to z\\ x\in G_i}} w(x) \qquad \text{(limiting value from the interior } G_i\text{)},\\ w^e(z) \;&:=\; \lim_{\substack{x\to z\\ x\in G_e}} w(x) \qquad \text{(limiting value from the exterior } G_e\text{)},\end{aligned} \tag{17}$$

we have the following continuity and jump relations (see Ladyzhenskaya [2, p. 56] in the case $n = 3$).

Lemma 3. *Let $\Psi \in C^0(\Gamma)$ and let $E_n^\bullet\Psi, D_n^\bullet\Psi, H_n^\bullet\Psi$ be the potentials defined by (12), (13), and (14). Then:*

$$(E_n^\bullet\Psi)^i \;=\; (E_n^\bullet\Psi) \;=\; (E_n^\bullet\Psi)^e, \tag{18}$$
$$(D_n^\bullet\Psi)^i - D_n^\bullet\Psi \;=\; +\tfrac{1}{2}\Psi \;=\; D_n^\bullet\Psi - (D_n^\bullet\Psi)^e, \tag{19}$$
$$(H_n^\bullet\Psi)^i - H_n^\bullet\Psi \;=\; -\tfrac{1}{2}\Psi \;=\; H_n^\bullet\Psi - (H_n^\bullet\Psi)^e. \tag{20}$$

2. Boundary Integral Equations

In the case $n = 2$ in $x \in G_e$ we choose the potential ansatz $(\eta, \alpha \in \mathbb{R})$

$$\binom{u}{p}(x) \;=\; -4\pi(E_2 a_\infty)(x) + (D_2\Psi)(x) - \eta(E_2 M_2 \Psi)(x) - \alpha \int\limits_\Gamma \binom{\Psi}{0}\, do. \tag{21}$$

Here $\Psi \in C^0(\Gamma)$ is an unknown source density, and the constant $a_\infty \in \mathbb{R}^2$ has to be selected identically to the prescribed constant a_∞ from the condition (6). The projector $M_n : C^0(\Gamma) \to C^0(\Gamma)$ is defined by

$$\Psi \to M_n\Psi \;:=\; \Psi - \Psi_M \tag{22}$$

with the surface mean value

$$\Psi_M \;:=\; \tfrac{1}{|\Gamma|}\int\limits_\Gamma \Psi\, do, \qquad |\Gamma| \;=\; \int\limits_\Gamma 1\, do.$$

In the case $n \geq 3$ a simpler formulation is sufficient. Here we choose in $x \in G_e$ $(\eta \in \mathbb{R})$ the potential ansatz

$$\binom{u}{p}(x) \;=\; (D_n\Psi)(x) - \eta(E_n\Psi)(x). \tag{23}$$

Using the continuity and jump relations of the potentials according to Lemma 3, now the following systems of boundary integral equations result from the formulations (21) and (23), respectively:

$$\Phi + 4\pi E_2 a_\infty \;=\; K_2\Psi \;:=\; \Big(-\tfrac{1}{2}I_2 + D_2^\bullet - \eta E_2^\bullet M_2 - \alpha|\Gamma|(I_2 - M_2)\Big)\Psi, \tag{24}$$
$$\Phi \;=\; K_n\Psi \;:=\; (-\tfrac{1}{2}I_n + D_n^\bullet - \eta E_n^\bullet)\Psi, \quad n \geq 3. \tag{25}$$

Here $\Phi \in C^0(\Gamma)$ is the prescribed boundary value. The solvability of these systems in $C^0(\Gamma)$ (see also [1] in the case $n = 2$) is described by the following theorem.

Theorem 4. *Let $\Phi \in C^0(\Gamma)$. Then:*

1. Let $a_\infty \in \mathbb{R}^2$ *be given. Then for* $\eta > 0$, $\alpha \neq 0$ *there exists exactly one solution* $\Psi = (\Psi_1, \Psi_2) \in C^0(\Gamma)$ *of the boundary integral equations' system (24).*

2. Let $3 \leq n \in \mathbb{N}$. *Then for* $\eta > 0$ *there exists exactly one solution* $\Psi = (\Psi_1, \dots, \Psi_n) \in C^0(\Gamma)$ *of the boundary integral equations' system (25).*

Let us now summarize the results.

Theorem 5. *Let* $G_e \subset \mathbb{R}^n$ $(n \geq 2)$ *be an exterior domain with boundary* $\Gamma \in C^2$. *Let* $\Phi \in C^0(\Gamma)$ *and in the case* $n = 2$ *also* $a_\infty \in \mathbb{R}^2$ *be given. Let* $\Psi \in C^0(\Gamma)$ *be the uniquely determined solution of the boundary integral equations (24) for* $n = 2$ *or (25) for* $n \geq 3$, *according to Theorem 4* $(\eta > 0,\ \alpha \neq 0)$. *Then (21) or (23), respectively, represents a solution* $u \in C^\infty(G_e) \cap C^0(\overline{G_e})$, $p \in C^\infty(G_e)$ *to the Stokes equations which satisfies the decay conditions from Lemma 1.*

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