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STOCHASTIC OPTIMIZATION AND UNCERTAINTY: AN EXAMPLE

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Abstract: R.T. Rockafellar's summary of acceptable approaches to stochatic optimization does not allow for uncertainty in F.H. Knight's sense, i.e., parameter variability with no distribution. A two level decision model, with the upper level decision model based on second order statistics of system performance and the lower level decision model a stochastic optimization problem, offers an opportunity for introducing Knightian uncertainty into the general stochastic optimization problem. We will illustrate the ideas with a portfolio selection example.

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1. Introduction

In previous work we have outlined a framework for multi-level decisions and coordination of multiple decision makers in an environment of uncertainty and risk [2]. The upper level decision models are based on second order statistics of system perfor-mance and lowest level decision problems are left unmodeled. We are concerned here with the application of Rockafellar's insight into stochastic optimization [4] for lowest level physics based models to a two-level decision problem with both uncertainty and risk

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For F.H. Knight [1] risk is associated with random variability with a distribution and uncertainty is variability without a distribution. Two distinct approaches for modeling uncertainty dominate [5]. In the Bayesian approach every source of variability has a distribution, perhaps uncertain. This approach is favored in the OR community since expressions of preference imply likelihood. In the "fuzzy set" approach uncertainty or lack of knowledge does not mean complete lack of knowledge. We always know something which can be captured with measures of belief. There is a third approach. Uncertainties are modeled as independent variables and performance is modeled as a random function of the uncertainties. Further, risk is defined as the probability of undesirable performance.

We employ a top/down system modeling/decomposition and bottom/up decision making. The basis of task/method decomposition is that methods are collections of tasks. This is not a physics based model decomposition, i.e., model components are tasks rather than physical components. Decisions are based on method performance. The goal of each decision maker is the best method of accomplishing his/her task. Method performances at a given level for different tasks may be correlated but there is no relationship between expected performances. The models are information models as opposed to flow models, i.e., there are no conservation rules.

Assume all random processes or fields modeling performance are normal. Complete descriptions are available in terms of means and covariances of performances. Estimates of second order statistics are available at the lowest level from data, simulations, or assumptions. We will discuss the interface between the lowest and the next higher level models later. Estimates are available at higher levels from lower level models.

Lowest level decision problems are stochastic optimization problems with uncertainty and risk. The models are physics based and of two types: either dynamic process models or portfolio problems.

2. A Two Level Portfolio Problem

The simpliest portfolio problem is an upper level decision problem. The optimization problem involves only finitely many alternatives using second order statistics. We do not need to resolve a mathematical programming problem. In a two level portfolio problem the lowest level problem will be a mathematical programming problem. We will begin by discussing the decision problem from the lowest level perspective.

2.1. An Example of a Lowest Level Portfolio Problem

An example serves to introduce all of the important concepts. There could be several criteria (only one for a simple example).

F: total yield on investment

There could be several uncertainties (only one for this example).

u: the rate of increase in the price of light crude

Contingencies for planning (maybe three). Contingencies in the example are transforming adaptations by society: the way we live, work, etc.

C_1: economic contraction

C_2: slow growth

C_3: current expectations for growth

$$prob(C_1) + prob(C_2) + prob(C_3) = 1\,.$$

Distributions of contingencies depend on the uncertainty (a key modeling decision). The probabilities of contingencies on the scaled rate of increase in the price of light crude u, $0 \leq u \leq 2/3$, are

$$\text{probability of contingencies: } (1/3,\, u,\, 2/3 - u)\,.$$

Finally, asset performance is contingency dependent. For a given contingency, the performance is deterministic. Consider the following three asset performances (total yield):

$$F_1(C) = \begin{cases} 1/2 & C = C_1 \\ 1/2 & C = C_2 \\ 1/2 & C = C_3 \end{cases} \qquad F_2(C) = \begin{cases} -1/2 & C = C_1 \\ 1 & C = C_2 \\ 1 & C = C_3 \end{cases} \qquad F_3(C) = \begin{cases} 1 & C = C_1 \\ 1 & C = C_2 \\ -1/2 & C = C_3 \end{cases}$$

The three assets are characterized, respectively, as safe, conservative, and aggressive. The uncertainty u must be passed down from the higher level. The F_is and C_is are introduced at the lowest level. We do not expect the higher level decision maker to understand the model. We assume there is a unit cost for each asset, i.e., the cost of one unit of the i-th asset is c_i. The stochastic optimization problem is

$$\begin{aligned} \text{minimize} \quad &: \quad X = -xF_1(\cdot,\, u) - yF_2(\cdot,\, u) - zF_3(\cdot,\, u) \\ \text{subject to} \quad &: \quad x\,c_1 + y\,c_2 + z\,c_3 = M \end{aligned}$$

$$x \geq 0,\ y \geq 0,\ \text{and } z \geq 0\,.$$

A decision based on the expected yield does not make use of all of the available information.

Rockafellar does not allow for uncertainty in our sense. He develops conditions for coherent measures of risk which permit the existence of an optimization structure. We will use a particular coherent measure of risk, namely, the conditional value at risk. The conditional value at risk of $-X(\cdot,\, u)$ is the expectation of $-X(\cdot,\, u)$ in the conditional distribution of its upper α-tail, i.e., the outcomes $-X(\cdot,\, u)$ in the upper part of the range of $-X(\cdot,\, u)$ for which the probability is $1-\alpha$. The conditional value at risk is conveniently equal to

$$\min_{H \in \mathcal{R}} \{H + (1-\alpha)^{-1} E[\max(0,\, -X(\cdot,\, u) - H)]\,.$$

A probabilistic interpretation (risk) is the following: "The conditional value at risk of $-X(\cdot,\, u) \leq B$" means not merely that $-X(\cdot,\, u) \leq B$ at least $100\alpha\%$ of the time, but that the average of theworst $100(1-\alpha)\%$ of all possible outcomes will be $\leq B$.

Suppose that M is a positive number, $0 < \alpha < 1$, H is a number, and u is a fixed value of the uncertainty. Assume that $c_1 = c_2 = c_3 = 1$. Let $S = \{(x,\, y,\, z) \,|\, 0 \leq x,\, y,\, z \leq M \text{ and } x+y+z = M\}$. The optimization problem becomes find $(x,\, y,\, z,\, H) \in S \times \mathcal{R}$ to minimize the following functional. Assume $\alpha = 0.15$.

$$\begin{aligned} H + \max(0,\, (-x/2 + y/2 - z)/3 + (-x/2 - y - z)u \\ + (-x/2 - y + z/2)(2/3 - u) - H)/0.85\,. \end{aligned}$$

2.2. Our Modeling/Decision Methodology

The higher level decision maker passes down to the lowest level a partition $0 = u_1 < u_2 < \cdots < u_n = 2/3$ of the interval of uncertainty $[0,\, 2/3]$. The lower level decision maker resolves n mathematical programming problems. Let $(x_i,\, y_i,\, z_i,\, H_i)$ be the solution of the ith mathematical programing problem with $u = u_i$. In Table 1, the uncertainty increases from 0 to 2/3. The triple $(x,\, y,\, z)$ represents investment intensities. Out of the $n = 129$ portfolios constructed, only six were different.

For $i = 1,\, 2,\, \ldots,\, n$, assume $(x,\, y,\, z)$ are the investment intensities for $u = u_i$. Let

$$\bar{\mu}_i(u_j) = (x/2 - y/2 + z)/3 + (x/2 + y + z)u_j + (x/2 + y - z/2)(2/3 - u_j)$$

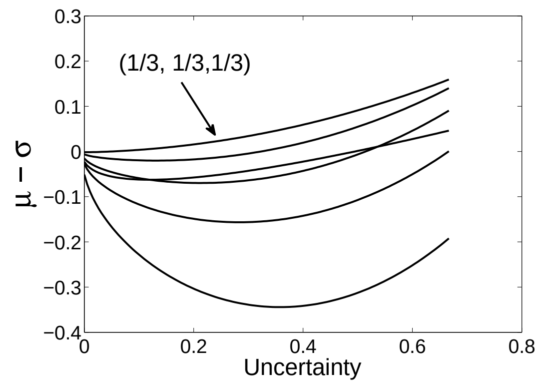

Figure 1: $\mu - \sigma$ for the six portfolios

and

$$\bar{\sigma}_i^2(u_j) = (x/2 - y/2 + z - \bar{\mu}(u_j))^2/3 + \\ (x/2 + y + z - \bar{\mu}(u_j))^2 u_j + (x/2 + y - z/2 - \bar{\mu}(u_j))^2 (2/3 - u_j)\,,$$

for $i,\, j = 1,\, 2,\, \ldots,\, n$.

2.3. Resolving the Problem

Optimizing stochastic systems requires the introduction of deterministic decision variables or surrogates. The goal is a preferred balance of expected payout (yield) and risk, the probability of an unfavorable outcome. $Y(u) = \int_0^u EX(v)\,dv + \int_0^u X(v) - EX(v)\sqrt{dv}$ is normal. $\mu(u) = EY(u) = \int_0^u \bar{\mu}(v)\,dv$ and $\sigma^2(u) = var(Y(u)) = \int_0^u \bar{\sigma}^2(v)\,dv$ An unfavorable outcome is $Y(u) \leq \mu(u) - \alpha\sigma(u)$, α a fixed real number. For the simple portfolio problem we can use $\mu - \alpha\sigma$ as a decision variable. Let $\mu_i(u_j) = \sum_{k=1}^{j+1} \bar{\mu}_i(u_k)/n$ and $\sigma_i^2(u_j) = \sum_{k=1}^{j+1} \bar{\sigma}_i^2(u_k)/n$ for $i,\, j = 1,\, 2,\, \ldots,\, n-1$. Figure 1 graphs the decision variable $\mu - \sigma$ for the six portfolios arrived at by resolving the programming problems with the uncertainty in turn fixed $u = u(i)$, for $i = 1,\, \ldots,\, 129$.

3. Conclusions

The “integrated” performance process is normal, the starting point for upper level model building. If $\bar{\mu}$ and $\bar{\sigma}^2$ are estimated from data μ and σ^2 will be smoother and facilitate analysis. Integration is an “averaging” process which also simplifies analysis. The balanced portfolio (1/3, 1/3, 1/3) takes into account performance over all possible uncertainty values.

Restricting mathematical programming to the lowest level permits a meaningful introduction of Knightian uncertainty. Upper level models permit a realistic risk analysis.

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