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# EXTERNAL POLYHEDRAL ESTIMATES FOR REACHABLE SETS OF LINEAR DISCRETE-TIME SYSTEMSWITH INTEGRAL BOUNDS ON CONTROLS

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**Abstract:** The approach for estimating the reachable sets $\mathcal{X}[k]$ of the linear discrete-time systems with integral bounds on controls is presented. It is based on considering reachable sets $\mathcal{Z}[k]$ in the "extended" phase space and allows to construct estimates for systems without and with state constraints. We construct the external estimates of $\mathcal{Z}[k]$ in the form of special polytopes. The specific cross-sections of them provide the parallelepiped-valued estimates of the reachable sets $\mathcal{X}[k]$ in the "initial" phase space. The whole families of estimates are introduced. Evolution of estimates is determined by recurrence relations. The families of touching estimates which ensure the exact representations of the reachable sets are described for systems without state constraints. The novel family of estimates for time-invariant systems is introduced. It is efficient for systems without and with small number of state constraints. The results of numerical simulations are presented.

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## 1. Introduction

The solution of many problems of the theory of control under uncertainty in a

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"guaranteed formulation" is based on the investigation of trajectory tubes [6]. Different numerical methods were developed for approximating reachable sets and trajectory tubes of systems with geometrical bounds on controls. Some of them are based on approximations of sets by domains of some fixed shape such as ellipsoids, parallelepipeds, zonotopes (see, for example, [6, 7, 1, 5, 2, 3] and references given there). There are few works devoted to approximating reachable sets of systems with integrally bounded controls (see [8, 4] and references therein). The aim of the paper is to develop the techniques for constructing approximations (estimates) of reachable sets to linear discrete-time systems with integrally bounded controls. Such systems may appear, in particular, as a result of approximation of the important class of linear differential systems with impulse controls. We follow the approach [6, 7, 5] and construct families of parallelepiped-valued estimates.

## 2. Problem Formulation

Consider a controlled system

$$x[j] = A[j]x[j-1] + B[j]u[j] + v[j], \quad j = 1, 2, \ldots, N; \tag{1}$$

$$x[0] \in \mathcal{X}_0 \subset \mathbb{R}^n; \quad \sum_{j=1}^{N} \|u[j]\|_\infty \le \mu_0; \tag{2}$$

$$u[j] \in \mathcal{K}[j] \subseteq \mathbb{R}^r, \quad j = 1, 2, \ldots, N, \tag{3}$$

with given matrices $A[j] \in \mathbb{R}^{n\times n}$ ($\det A[j] \neq 0$), $B[j] \in \mathbb{R}^{n\times r}$, inputs $v[j] \in \mathbb{R}^n$, initial set $\mathcal{X}_0$ and with controls $u[j] \in \mathbb{R}^r$ subjected to the given integral (2) and hard (3) constraints, where $\mathcal{K}[j]$ are convex closed cones in $\mathbb{R}^r$ ($r \le n$), $\mu_0 \ge 0$, $\|u\|_\infty = \max_{1\le i\le r} |u_i|$. These relations may be complemented by state constraints

$$x[k_i] \in \mathcal{Y}[k_i] \subseteq \mathbb{R}^n, \quad i = 1, 2, \ldots, N_c \quad (k_{N_c} \le N). \tag{4}$$

The *reachable set* $\mathcal{X}[k]$ for system (1)–(3) ((1)–(4)) at time $k \in \{1, 2, \ldots, N\}$ is a set of those points $x \in \mathbb{R}^n$, for each of which there exists a pair $\{x[0], u[\cdot]\}$ that satisfies (2)–(3) and generates a solution $x[\cdot]$ of (1) that satisfies $x[k] = x$ (and (4) for all $k_i \le k$). Set-valued map $\mathcal{X}[k]$, as a function of $k$, defines a so-called *trajectory tube* $\mathcal{X}[\cdot]$. If (4) are generated by a measurement equation

$$y[k_i] = G[k_i]\, x[k_i] + \eta[k_i], \quad \eta[k_i] \in \Theta[k_i] \subset \mathbb{R}^m,$$

with unknown but bounded error, then $\mathcal{X}[k]$ are also referred to as the *informational domains* [6].

**Assumption.** We presume $\mathcal{X}_0$ to be a parallelepiped $\mathcal{X}_0 = \mathcal{P}_0$, the sets

$\mathcal{R}[j] = \mathcal{C} \cap \mathcal{K}[j]$, where $\mathcal{C}$ is the unit cube with the center at origin, to be parallelepipeds, and $\mathcal{Y}[j]$ to be zones.

By the *parallelepiped* $\mathcal{P}(p, P, \pi) \subset \mathbb{R}^n$ we mean a set such that $\mathcal{P} = \mathcal{P}(p, P, \pi) = \{x \mid x = p + \sum_{i=1}^{n} p^i \pi_i \xi_i, \quad |\xi_i| \leq 1\}$. Here $p \in \mathbb{R}^n$; $P = \{p^i\} \in \mathcal{M}_*^{n\times n}$, $\mathcal{M}_*^{n\times n} = \{P \in \mathbb{R}^{n\times n} | \det P \neq 0, \|p^i\| = 1\}$[1]; $\pi \in \mathbb{R}^n$, $\pi \geq 0$ (vector inequalities are understood componentwise). It may be said that $p$ determines the center of the parallelepiped, $P$ — orientation matrix, $p^i$ — "directions" and $\pi_i$ — values of its "semi-axes". The parallelepiped may be represented in other form: $\mathcal{P} = \mathcal{P}(P, \gamma^{(-)}, \gamma^{(+)}) = \{x \,|\, \gamma^{(-)} \leq P^{-1}x \leq \gamma^{(+)}\}$, $\gamma^{(\pm)} = P^{-1}p \pm \pi$.

By the *zone* $S = \mathcal{S}(c, S, \sigma, m) \subset \mathbb{R}^n$ we mean the intersection of $m \leq n$ *strips*: $S = \bigcap_{i=1}^{m} \Sigma^i$, $\Sigma^i = \{x \,|\, |{s^i}^\top x - c_i| \leq \sigma_i\}$. Here $c \in \mathbb{R}^m$, $S = \{s^i\} \in \mathbb{R}^{n\times m}$, vectors $s^i$ are linear independent, $\sigma \in \mathbb{R}^m$, $\sigma \geq 0$, $\top$ is the transposition symbol.

We call a parallelepiped $\mathcal{P}$ *external estimate for* $\mathcal{Q} \subset \mathbb{R}^n$ if $\mathcal{P} \supseteq \mathcal{Q}$. We call an external estimate of $\mathcal{Q} \subset \mathbb{R}^n$ *tight (in direction $l$)* if there exists $l \in \mathbb{R}^n$ such that $\rho(\pm l|\mathcal{P}) = \rho(\pm l|\mathcal{Q})$, where $\rho(l|\mathcal{Q}) = \sup\{(x, l)|x \in \mathcal{Q}\}$ is the support function of $\mathcal{Q}$. We call a parallelepiped $\mathcal{P}(p, P, \pi)$ *touching external estimate for* $\mathcal{Q}$ if it is tight estimate in $n$ specified directions $l^i = {P^{-1}}^\top \mathrm{e}^i$, $i = 1, 2, \ldots, n$, where $\mathrm{e}^i = (0, \ldots, 0, 1, 0, \ldots, 0)^\top$ is the unit vector oriented along the axis $0x_i$ (the unit stands at $i$-position).

Consider the following *problems*:

— Find some external estimates $\mathcal{P}^+[k] = \mathcal{P}(p^+[k], P^+[k], \pi^+[k])$ for $\mathcal{X}[k]$. Moreover, introduce some families of such estimates.

— Introduce some families of touching estimates $\mathcal{P}^+[k]$.

— Introduce some families of estimates $\mathcal{P}^+[k]$ which ensure exact representations of reachable sets through the intersection of units of the family: $\mathcal{X}[k] = \bigcap \mathcal{P}^+[k]$.

For finding $\mathcal{X}[k]$ it is convenient to introduce reachable sets $\mathcal{Z}[k]$ of states $z = \{x, \mu\} = (x^\top, \mu)^\top \in \mathbb{R}^{n+1}$ for system (1), (3), (4), (5)-(7):

$$\mu[j] = \mu[j-1] - \|u[j]\|_\infty, \quad j = 1, 2, \ldots, N; \tag{5}$$

$$\mu[j] \geq 0, \quad j = 1, 2, \ldots, N; \tag{6}$$

$$z[0] = \{x[0], \mu[0]\} \in \mathcal{Z}_0 = \mathcal{P}_0 \times [0, \mu_0] \tag{7}$$

in the "extended" phase space, where $\mu$ corresponds to a current control stock.

It is convenient to seek the reachable sets $\mathcal{Z}[k]$ of this system in the form of the union of their $\mu$-cross-sections $\mathcal{X}(\mu, k)$: $\mathcal{Z}[k] = \bigcup_{0 \leq \mu \leq \mu^+[k]} \{\mathcal{X}(\mu, k), \mu\}$.

[1]The normality condition $\|p^i\| = 1$ may be omitted to simplify formulas.

It may be verified [4] that the reachable sets $\mathcal{X}[k]$ coincide with the "lower" cross-sections of $\mathcal{Z}[k]$ corresponding to the value $\mu = 0$: $\mathcal{X}[k] = \mathcal{X}(0, k)$.

Sets $\mathcal{Z}[k]$ (unlike $\mathcal{X}[k]$) satisfy the upper semigroup property [6, 4]. It permits to obtain some recurrence relations [4] for $\mathcal{Z}[k]$ which involve the following operations with the sets of the form $\mathcal{Z} = \bigcup_{0\le\mu\le\mu^{\rm t}}\{\mathcal{X}(\mu), \mu\} \subseteq \mathbb{R}^{n+1}$: $A \bigodot \mathcal{Z} = \bigcup_{0\le\mu\le\mu^{\rm t}}\{A\mathcal{X}(\mu), \mu\}$, $\forall A \in \mathbb{R}^{n\times n}$; $\mathcal{Z} \bigoplus v = \bigcup_{0\le\mu\le\mu^{\rm t}}\{\mathcal{X}(\mu) + v, \mu\}$, $\forall v \in \mathbb{R}^n$; $\mathcal{Z} \circledcirc \mathcal{Y} = \bigcup_{0\le\mu\le\mu^{\rm t}}\{\mathcal{X}(\mu) \cap \mathcal{Y}, \mu\}$, $\forall \mathcal{Y} \subseteq \mathbb{R}^n$; $\mathcal{Z} \uplus \mathcal{R} = \tilde{\mathcal{Z}} = \bigcup_{0\le\mu\le\mu^{\rm t}}\{\tilde{\mathcal{X}}(\mu), \mu\}$, where $\tilde{\mathcal{X}}(\mu) = \bigcup_{\mu\le\zeta\le\mu^{\rm t}}(\mathcal{X}(\zeta) + (\zeta - \mu)\mathcal{R})$, $\forall \mathcal{R} \subset \mathbb{R}^n$.

We will find external estimates for $\mathcal{Z}[k]$ in the form of polytopes

$$\mathit{\Pi} = \mathit{\Pi}(\{\mathcal{P}^{\rm b}, 0\}, \{\mathcal{P}^{\rm t}, \mu^{\rm t}\}) \subset \mathbb{R}^{n+1}$$

of some specific form ("*$\mathit{\Pi}$-politopes*" [4]) which are defined by their "lower" and "upper" cross-sections through the operation of convex hull, where the both mentioned cross-sections are the parallelepipeds with the identical orientation matrices: $\mathit{\Pi} = \mathrm{co}\,(\{\mathcal{P}^{\rm b}, 0\} \cup \{\mathcal{P}^{\rm t}, \mu^{\rm t}\})$, where $\mathcal{P}^{\rm b} = \mathcal{P}(p^{\rm b}, P, \pi^{\rm b})$, $\mathcal{P}^{\rm t} = \mathcal{P}(p^{\rm t}, P, \pi^{\rm t})$, $\mu^{\rm t} \ge 0$. We call a $\mathit{\Pi}$-politope the *touching external estimate for* $\mathcal{Z} \subset \mathbb{R}^{n+1}$ if $\mathcal{Z} \subseteq \mathit{\Pi}$ and all $\mu$-cross-sections of $\mathit{\Pi}$ are touching estimates for $\mu$-cross-sections of $\mathcal{Z}$.

### 3. Polyhedral Estimates for $\mathcal{Z}[k]$ and $\mathcal{X}[k]$

Calculating estimates for $\mathcal{Z}[k]$ is based on operating with $\mathit{\Pi}$-politopes. Sets $\mathit{\Pi} \bigoplus v$, $A \bigodot \mathit{\Pi}$ are $\mathit{\Pi}$-politopes $\forall v \in \mathbb{R}^n$, $\forall A \in \mathbb{R}^{n\times n}$, $\det A \ne 0$. The result of operationes $\uplus$ and $\circledcirc$ will be estimated from outside. At that primary polyhedral estimates for results of set operations with parallelepipeds in $\mathbb{R}^n$ are used.

The touching external parallelepiped-valued estimate for $Q \subset \mathbb{R}^n$ with a given orientation matrix $V$ is determined by the formula [3] $\boldsymbol{P}_V^+(\mathcal{Q}) = \mathcal{P}(V, \gamma^{(-)}, \gamma^{(+)})$, $\gamma_i^{(\pm)} = \pm\rho(\pm {V^{-1}}^{\top} \mathrm{e}^i | \mathcal{Q})$. In particular, $\boldsymbol{P}_V^+(\sum_{k=1}^2 \mathcal{P}(p^k, P^k, \pi^k)) = \mathcal{P}(\sum_{k=1}^2 p^k, V, \sum_{k=1}^2 (\mathrm{Abs}\,(V^{-1}P^k))\pi^k)$ (where $\mathrm{Abs}\,A = \{|a_i^j|\}$ for $A = \{a_i^j\}$) and $\boldsymbol{P}_V^+(\mathcal{P}^1 \cup \mathcal{P}^2)$ can be calculated using the formula $\rho(l | \mathcal{P}^1 \cup \mathcal{P}^2) = \max_{1\le k\le 2} \rho(l | \mathcal{P}^k)$.

It may be verified that the touching estimate $\mathbf{\Pi}_V^+(\mathcal{Z}) = \mathit{\Pi}(\{\mathcal{P}^{+\rm b}, 0\}, \{\mathcal{P}^{+\rm t}\}, \{\mu^{+\rm t}\})$ for $\mathcal{Z} = \mathit{\Pi} \uplus \mathcal{P} \subset \mathbb{R}^{n+1}$ is determined by formulas $\mathcal{P}^{+\rm t} = \boldsymbol{P}_V^+(\mathcal{P}^{\rm t})$, $\mu^{+\rm t} = \mu^{\rm t}$, $\mathcal{P}^{+\rm b} = \boldsymbol{P}_V^+(\mathcal{P}^{\rm b} \cup (\mathcal{P}^{\rm t} + \mu^{\rm t}\mathcal{P}))$.

External estimates $\mathbf{\Pi}_{\rm par}^+(\mathcal{Z})$ for $\mathcal{Z} = \mathit{\Pi} \circledcirc \Sigma$, where $\mathit{\Pi}$ is a $\mathit{\Pi}$-polytope, $\Sigma$ is a strip, when several special orientation matrices $V \in \mathcal{V}(\mathit{\Pi}, \Sigma)$ are used, also can be constructed by some explicit formulas (see [4] for details). Here par denote the list of parameters: par $= \{V, g^{(-)}, g^{(+)}\}$, $\mathcal{V}(\mathit{\Pi}, \Sigma)$ is the set of $n+1$ admissible orientation matrices $V \in \mathbb{R}^{n\times n}$, vectors $g^{(\pm)} \in \mathbb{R}^n$ are admissible values of parameters which determine the incline of "side faces" of $\mathbf{\Pi}^+_{\mathrm{par}}(\mathcal{Z})$.

For the case without state constraints consider the family $\mathfrak{P}^+$ of tubes $\mathit{\Pi}^+[\cdot]$ satisfying

$$\begin{gathered}\mathit{\Pi}^+[k] = \mathbf{\Pi}^+_{P^+[k]}((A[k]\bigodot \mathit{\Pi}^+[k-1]\bigoplus v[k]) \uplus B[k]\mathcal{R}[k]),\\ k=1,2,\ldots,N, \qquad \mathit{\Pi}^+[0] = \mathbf{\Pi}^+_{P^+[0]}(\mathcal{Z}_0),\end{gathered} \tag{8}$$

where $\mathcal{R}[k] = \mathcal{C} \cap \mathcal{K}[k] \subset \mathbb{R}^r$, $\mathcal{C} = \mathcal{P}(0, I, \mathrm{e})$, $I$ is the unit matrix, $\mathrm{e} = (1, 1, \ldots, 1)^\top$. Here the orientation matrices $P^+[\cdot]$ serve as a parameter of the family. Let us also introduce the subfamily $\mathfrak{P}^{1+} \in \mathfrak{P}^+$ for which $P^+[\cdot]$ satisfy

$$P^+[k] = A[k]P^+[k-1],\ k{=}1,\ldots,N,\ P^+[0] = P \ \ (P \in \mathcal{M}^{n\times n}_*).$$

**Theorem 1.** (see [4]) *Let $\mathcal{X}[k]$ and $\mathcal{Z}[k]$ be reachable sets for system (1)–(3) and (1), (3), (5)–(7) respectively. If $\mathit{\Pi}^+[\cdot] = \mathit{\Pi}(\{\mathcal{P}^{+\mathrm{b}}[\cdot], 0\}, \{\mathcal{P}^{+\mathrm{t}}[\cdot], \mu^{+\mathrm{t}}[\cdot]\}) \in \mathfrak{P}^+$, then $\mathit{\Pi}$-polytopes $\mathit{\Pi}^+[k]$ and parallelepipeds $\mathcal{P}^{+\mathrm{b}}[k]$ are the external estimates for $\mathcal{Z}[k]$ and $\mathcal{X}[k]$ respectively: $\mathcal{Z}[k] \subseteq \mathit{\Pi}^+[k]$, $\mathcal{X}[k] \subseteq \mathcal{P}^{+\mathrm{b}}[k]$, $k = 1, 2, \ldots, N$, whatever are $P^+[k] \in \mathcal{M}^{n\times n}_*$, $k = 1, 2, \ldots, N$. If $\mathit{\Pi}^+[\cdot] \in \mathfrak{P}^{1+}$ is the tube corresponding to an arbitrary matrix $P \in \mathcal{M}^{n\times n}_*$ in (9), then $\mathit{\Pi}^+[k]$, $\mathcal{P}^+[k]$ are touching estimates for $\mathcal{Z}[k]$, $\mathcal{X}[k]$ respectively, and $\mathcal{Z}[k] = \bigcap\{\mathit{\Pi}^+[k] |\ P \in \mathcal{V}^0\}$, $\mathcal{X}[k] = \bigcap\{\mathcal{P}^{+\mathrm{b}}[k] |\ P \in \mathcal{V}^0\}$. Here $\mathcal{V}^0$ denotes an arbitrary set of matrices $P \in \mathcal{M}^{n\times n}_*$ with the property that, for each nonzero vector $l \in \mathbb{R}^n$, there is a matrix $P$ in this set such that $l$ is collinear to some column of $(P^{-1})^\top$.*

If the system is time-invariant ($A[k] \equiv A$, $B[k] \equiv B$, $\mathcal{K}[k] \equiv \mathcal{K}$), then it may be verified, using the idea from Girard et al [2], that $\mathcal{Z}[k]$ also satisfy some another system of recurrence relations where the other (in comparison with [4]) order of set operations is used. Then let us introduce the family $\mathfrak{P}^{2+} \notin \mathfrak{P}^+$ of tubes $\mathit{\Pi}^+[\cdot]$:

$$\begin{aligned}&\mathit{\Pi}^+[k] = \mathbf{\Pi}^+_P(\mathit{\Pi}[k] \uplus \mathcal{P}^1[k]),\ k=1,2,\ldots,N;\\ &\mathit{\Pi}[k] = A\bigodot \mathit{\Pi}[k-1]\bigoplus v[k],\ k=1,2,\ldots,N, \quad \mathit{\Pi}[0] = \mathcal{Z}_0;\\ &\mathcal{P}^1[k] = \boldsymbol{P}^+_P(\mathcal{P}^1[k-1] \cup \mathcal{P}^2[k-1]), \qquad \mathcal{P}^1[0] = \mathcal{P}(0, I, 0);\\ &\mathcal{P}^2[k] = A\mathcal{P}^2[k-1],\ k=1,2,\ldots,N, \qquad\quad \mathcal{P}^2[0] = B\mathcal{R},\end{aligned} \tag{9}$$

where the constant orientation matrix $P$, $\det P{\neq}0$, is a parameter of the family.

**Theorem 2.** *Let $\mathcal{X}[k]$ be reachable sets for system (1)–(3) which is time-invariant and $\mathcal{Z}[k]$ be reachable sets for corresponding system (1), (3), (5)–(7). If $\mathit{\Pi}^+[\cdot] = \mathit{\Pi}(\{\mathcal{P}^{+\mathrm{b}}[\cdot], 0\}, \{\mathcal{P}^{+\mathrm{t}}[\cdot], \mu^{+\mathrm{t}}[\cdot]\}) \in \mathfrak{P}^{2+}$, then $\mathit{\Pi}^+[k]$, $\mathcal{P}^{+\mathrm{b}}[k]$ are touching estimates for $\mathcal{Z}[k]$, $\mathcal{X}[k]$ respectively, whatever is $P \in \mathcal{M}^{n\times n}_*$, and*

$\mathcal{Z}[k] = \bigcap\{\Pi^+[k] \mid P \in \mathcal{V}^0\}$, $\mathcal{X}[k] = \bigcap\{\mathcal{P}^{+\mathrm{b}}[k] \mid P \in \mathcal{V}^0\}$, *where* $\mathcal{V}^0$ *is the same as above.*

For the case under zone-valued state constraints with $\mathcal{Y}[k_i] = \bigcup_{\alpha=1}^m \Sigma^\alpha[k]$, let $\mathfrak{P}^+$ be the family of tubes $\Pi^+[\cdot]$ satisfying

$$\Pi^+[k] = \begin{cases} \Pi^{0+}[k], \text{ if } k{=}1,2,\dots,N, \text{ but } k{\neq}k_i,\ i{=}1,2,\dots,N_c, \\ \Pi^{m+}[k], \text{ for } k{=}k_i, \quad i{=}1,2,\dots,N_c; \end{cases}$$
$$\Pi^{\alpha+}[k] = \mathbf{\Pi}^+_{\mathrm{par}^{\alpha+}[k]}(\Pi^{\alpha-1,+}[k] \circledcirc \Sigma^\alpha[k]),\ \alpha{=}1,2,\dots,m, \text{ for } k{=}k_i; \tag{10}$$

$$\Pi^{0+}[k] = \mathbf{\Pi}^+_{P^{0+}[k]}((A[k]\bigodot\Pi^+[k{-}1]\bigoplus v[k]) \uplus B[k]\mathcal{R}[k]),$$
$$k{=}k_i{+}1, k_i{+}2, \dots, k_{i+1}, i{=}0,1\dots,N_c{-}1, \quad \Pi^{0+}[0]{=}\mathbf{\Pi}^+_{P^+[0]}(\mathcal{Z}_0). \tag{11}$$

Here $\mathrm{par}^{\alpha+}[k]$ denotes the list of additional (besides matrices $P^{0+}[k]$) parameters of the family: $\mathrm{par}^{\alpha+}[k] = \{P^{\alpha+}[k], g^{\alpha(-)}[k], g^{\alpha(+)}[k]\}$; $k_0 = 0$.

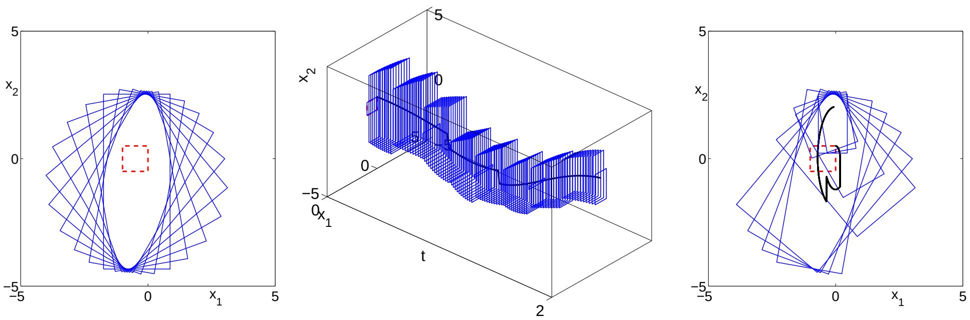
For time-invariant systems under zone-valued state constraints let $\mathfrak{P}^{2+}$ be the family of tubes $\Pi^+[\cdot]$ satisfying (10) and relations

$$\Pi^{0+}[k] = \mathbf{\Pi}^+_{P^{0+}[k_i]}(\Pi[k] \uplus \mathcal{P}^1[k]),\ k = k_i{+}1, k_i{+}2, \dots, k_{i+1};$$
$$\Pi[k] = A\bigodot\Pi[k{-}1]\bigoplus v[k] \quad (\text{for the same } k),\ \Pi[k_i] = \Pi^+[k_i];$$
$$\mathcal{P}^1[k] = \boldsymbol{P}^+_{P^{0+}[k_i]}(\mathcal{P}^1[k{-}1] \cup \mathcal{P}^2[k{-}1]), \qquad \mathcal{P}^1[k_i] = \mathcal{P}(0, I, 0);$$
$$\mathcal{P}^2[k] = A\mathcal{P}^2[k{-}1] \quad (\text{for the same } k), \qquad \mathcal{P}^2[k_i] = B\mathcal{R},$$

where $i = 0, 1\dots, N_c{-}1$, $k_0 = 0$, $\Pi^+[0] = \mathcal{Z}_0$.

**Theorem 3.** *Let* $\mathcal{X}[k]$ *and* $\mathcal{Z}[k]$ *be reachable sets for system (1)–(4) and (1), (3)–(7) respectively. If* $\Pi^+[\cdot] \in \mathfrak{P}^+$, *then* $\mathcal{Z}[k] \subseteq \Pi^+[k]$, $\mathcal{X}[k] \subseteq \mathcal{P}^{+\mathrm{b}}[k]$, $k{=}1, 2, \dots, N$, *whatever are* $P^{0+}[k] \in \mathcal{M}_*^{n\times n}$, $k{=}1, 2, \dots, N$, *and admissible values of parameters* $\mathrm{par}^{\alpha+}[k_i]$, $\alpha{=}1, 2, \dots, m$, $i = 1, 2, \dots, N_c$. *If system (1)–(3) is time-invariant and* $\Pi^+[\cdot] \in \mathfrak{P}^{2+}$, *then all* $\mathcal{Z}[k] \subseteq \Pi^+[k]$, $\mathcal{X}[k] \subseteq \mathcal{P}^{+\mathrm{b}}[k]$, *whatever are* $P^{0+}[k_i] \in \mathcal{M}_*^{n\times n}$, $i = 1, 2, \dots, N_c$, *and the rest admissible values of parameters* $\mathrm{par}^{\alpha+}[k_i]$ *(*$\alpha{=}1, 2, \dots, m$*).*

**Example.** Consider a system obtained by discretization of an oscillatory impulse differential one: $A[j] \equiv I + h_N A$, $h_N = \theta N^{-1}$, $\theta = 2$, $N = 200$, $B[j] \equiv B$, $A{=}\begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$, $B{=}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v[j] \equiv 0$, $\mu_0 = 2$, $\mathcal{K} = \mathbb{R}^1$, $\mathcal{X}_0 = \mathcal{P}(-0.5\mathrm{e}^1, I, 0.5\mathrm{e})$. Figure 1(a) shows $\mathcal{X}_0$ (in dash line) and several touching estimates $\mathcal{P}^+[N]$ for the reachable set $\mathcal{X}[N]$ for the case $\mathfrak{P}^{2+}$. In aggregate, they "outline" $\mathcal{X}[N]$. Figure 1(b) and (c) illustrate estimates from $\mathfrak{P}^{2+}$ for informational domains (of the same system) generated by measurements $y[k_i] = x_2[k_i]{+}\eta[k_i]$, $|\eta[k_i]| \le 0.1$, $k_i{=}25i$, $i{=}1, \dots, 7$, when the realization of $x[\cdot]$ corresponds to $x_0 = \mathrm{e}$, $u[\cdot]$ with $u[62] = u[112] = 0.5\mu_0$ and the rest $u[j] \equiv 0$, $\eta[k_i] = \pm 0.1$ (bang-bang




(a) (b) (c)

Figure 1:

type). Figure 1(b) shows some tube $\mathcal{P}^+[\cdot] \supset \mathcal{X}[\cdot]$. Figure 1(c) presents several estimates $\mathcal{P}^+[N]$ for $\mathcal{X}[N]$ which are obtained using tubes $\mathit{\Pi}^+[\cdot] \in \mathfrak{P}^{2+}$ and then are "improved" by intersecting them with the estimates for reachable sets of the system without state constraints. These estimates turn out to be smaller than those for the case without state constraints.

See [4] for some results of modelling estimates from $\mathfrak{P}^{1+}$, $\mathfrak{P}^+$.

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