International Journal of Pure and Applied Mathematics
Volume 50 No. 2 2009, 195-200

THE ELLIPSOIDAL ESTIMATES OF REACHABLE SETS OF IMPULSIVE UNCERTAIN SYSTEMS WITH COMBINED CONTROLS

Oksana G. Vzdornova
Institute of Mathematics and Mechanics
Ural Branch of Russian Academy of Sciences
16, S. Kovalevskaya Str., GSP-384, Ekaterinburg, 620219, RUSSIA
e-mail: vog@imm.uran.ru

Abstract: The present paper is devoted to the state estimation problem for impulsive control systems described by linear differential equations with combined control and with uncertain initial state.

AMS Subject Classification: 93B03, 93C05, 93C41
Key Words: impulsive control, reachable sets, estimation

1. Introduction

The state estimation problems for dynamic control systems with uncertain initial state were studied in [3, 4, 1]. We consider here the case of combined controls. The control is a pair of classical (measurable) control and the impulsive control function. We assume that both parts of admissible control are restricted by ellipsoids. Namely, the classical control should belong to a given finite-dimensional ellipsoid and there is also an additional constraint on the impulsive control part represented by a special restriction of ellipsoidal type. Such problems arise when the possibilities of control of impulsive dynamic system are non-even in different directions [6, 5].

The problem is studied under uncertainty conditions with set – membership description of uncertain initial states, which are taken to be unknown but bounded with given bounds. In this paper we use the well known results of pro-

Received: August 14, 2008

fessors A.B. Kurzhanski [3, 4] and F.L. Chernousko [1] devoted to ellipsoidal state estimation theory for dynamical systems with classical (measurable) controls.

We present here the method of constructing ellipsoidal estimates of reachable sets of dynamic control systems with combined control and with uncertain initial state that uses the special structure of linear control problem.

2. Problem Formulation

Consider a dynamic control system described by a differential equation with combined control:

$$dx = A(t)xdt + v(t)dt + du(t),\ x \in R^n,\ x(-0) = x_0,\ t \in [0;T]\,, \tag{1}$$

or in the integral form

$$x(t; v(\cdot), u(\cdot), x_0) = X(t)x_0 + \int_0^t X(t)X^{-1}(\tau)v(\tau)d\tau + \int_0^t X(t)X^{-1}(\tau)du(\tau). \tag{2}$$

Here we assume that $A(t)$ is a continuous $n \times n$ – matrix function on $[0, T]$, $X(t)$ is the fundamental matrix solution $\dot{X} = A(t)X$ $(X(0) = I)$. The control is a pair of classical (measurable) control $v(t)$ and the impulsive control function $u(t)$. The integrals in (2) are taken as the Lesbegue integral and as Riemann-Stieltjes integral relatively.

The control v belongs to a given finite-dimensional ellipsoid

$$v(t) \in \mathcal{E}(0, P) = \{z \in R^n |\ z'P^{-1}z \le 1\},\ t \in [0, T], \tag{3}$$

where P is a given symmetric positive definite $n \times n$ matrix. Denote by $\mathcal{V}$ the class of measurable functions $v(t)$ that satisfy (3).

Given an ellipsoid $\mathcal{E}_0 = \mathcal{E}(0, Q_0^{-1}) = \{l \in R^n |\ \ l'Q_0 l \le 1\}$, we introduce a special restriction $\mathcal{U}$ on the impulsive part $u(\cdot)$ of control functions in the space V^n of functions of bounded variations on $[0, T]$ [6]:

$$\mathcal{U} = \{u(\cdot) \in V^n |\ \int_0^T y(t)du(t) \le 1, \forall y(\cdot) \in C^n,\ y(t) \in \mathcal{E}_0,\ \forall t \in [0, T]\}, \tag{4}$$

where C^n is the space of continuous n-vector functions $y(\cdot)$, Q_0 is a given symmetric positive definite $n \times n$ matrix.

In particular, under such restriction vectors of impulsive jumps of controls $\Delta u = u(t_{i+1}) - u(t_i) \in \mathcal{U}$ have to lie in the ellipsoid $\mathcal{E}_0^* = \mathcal{E}(0, Q_0)$.

We will assume that the initial value $x(-0) = x_0$ for the system (1) is unknown but bounded with a given bound $x_0 \in \mathcal{X}_0 = \mathcal{E}(0, R)$, where R is a symmetric positive definite $n \times n$ matrix.

Denote

$$\mathcal{X}(t; \mathcal{X}_0) = \bigcup_{x_0 \in \mathcal{X}_0} \bigcup_{v \in \mathcal{V}} \bigcup_{u \in \mathcal{U}} \{x(t; v(\cdot), u(\cdot), x_0)\} \tag{5}$$

the reachable set of the impulsive differential system (1) from the initial set $\mathcal{X}_0$ at the instant t under restriction (3)-(4).

So the main problem considered in this paper is to find external and internal ellipsoidal estimates for reachable sets $\mathcal{X}(t; \mathcal{X}_0)$ of dynamical control system (1) with combined controls and with uncertain initial state basing on the special structure of the data $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{X}_0$.

3. Main Results: Ellipsoidal Estimates

In this section we apply the techniques of the ellipsoidal calculus to find the estimates for the $\mathcal{X}(t; \mathcal{X}_0)$. For simplicity and without loss of generality we will consider the case $t = T$.

Consider, at first, two auxiliary problems.

Problem 1. Find an external $\mathrm{E}_1^+ = \mathcal{E}(0, \mathrm{Q}_1^+)$ and internal $\mathrm{E}_1^- = \mathcal{E}(0, \mathrm{Q}_1^-)$ ellipsoidal estimates for the reachable set $\mathcal{X}_1(T; \mathcal{X}_0)$ of impulsive system

$$dx = A(t)xdt + du(t), \quad x(-0) \in \mathcal{X}_0, \quad t \in [0; T], u \in \mathcal{U}. \tag{6}$$

The solution of Problem 1 was considered in [6, 5]. In these papers in order to find ellipsoidal estimates we used the fact that the reachable set $\mathcal{X}_1(T; \mathcal{X}_0)$ of system (6) has the following forms:

$$\mathcal{X}_1(T, \mathcal{X}_0) = \bigcup_{x_0 \in \mathcal{X}_0} \bigcup_{u \in \mathcal{U}} \{X(T)x_0 + \int_0^T X(T)X^{-1}(\tau)du(\tau)\} \tag{7}$$

and that for any $\varepsilon > 0$ there exist $\delta > 0$ and finite set $T_\delta \subset [0, T]$ such that the inclusions are true

$$\mathcal{X}_0 + co(\bigcup_{\tau \in T_\delta} \mathcal{E}(0, Q_\tau)) \subset \mathcal{X}_1(T, \mathcal{X}_0) \subset \mathcal{X}_0 + co(\bigcup_{\tau \in T_\delta} \mathcal{E}(0, Q_\tau)) + \varepsilon S,$$

where $Q_\tau = X(T, \tau)Q_0X'(T, \tau)$. So, in the case of Problem 1 we need to construct the external and internal ellipsoidal estimates for the convex hull of the union of a family of ellipsoids. This problem does not arise in the case of estimating states of dynamic systems with controls of classical type [3] and is new for estimation problems in impulsive systems [6, 5].

Problem 2. Find an external $\mathrm{E}_2^+ = \mathcal{E}(0, \mathrm{Q}_2^+)$ and internal $\mathrm{E}_2^- = \mathcal{E}(0, \mathrm{Q}_2^-)$ ellipsoidal estimates for the reachable set $\mathcal{X}_2(T, \{0\})$ of system

$$\dot{x} = A(t)x + v, \quad x(0) = 0, \quad t \in [0; T], v \in \mathcal{V}. \tag{8}$$

In this case the reachable set $\mathcal{X}_2(T, \{0\})$ of system (8) has the form:

$$\mathcal{X}_2(T, \{0\}) = \mathcal{X}_2(T) = \bigcup_{v \in \mathcal{V}} \{ \int\limits_0^T X(T) X^{-1}(\tau) v(\tau) d\tau \}. \tag{9}$$

The solution of Problem 2 is done in the monographes [4, 1].

The following lemma is true.

Lemma. *The reachable set $\mathcal{X}(T, \mathcal{X}_0)$ of system (1) with constrains (3)-(4) has the form*

$$\mathcal{X}(T, \mathcal{X}_0) = \mathcal{X}_1(T, \mathcal{X}_0) + \mathcal{X}_2(T). \tag{10}$$

Proof. Formula (10) follows from the structure of the systems (1), (6), (8) and formulas (2), (5), (7), (9). □

Theorem. *The inclusions*

$$\mathcal{X}(T, \mathcal{X}_0) \subset \mathrm{E}_1^+ + \mathrm{E}_2^+ \subset \mathrm{E}^+ = \mathcal{E}(0, \mathrm{Q}^+), \tag{11}$$

$$\mathcal{E}(0, \mathrm{Q}^-) = \mathrm{E}^- \subset \mathrm{E}_1^- + \mathrm{E}_2^- \subset \mathcal{X}(T, \mathcal{X}_0), \tag{12}$$

are true, where

$$\mathrm{Q}^- = \mathrm{Q}_1^- + \mathrm{Q}_2^- + 2(\mathrm{Q}_1^-)^{1/2}((\mathrm{Q}_1^-)^{-1/2}\mathrm{Q}_2^-(\mathrm{Q}_1^-)^{-1/2})^{1/2}(\mathrm{Q}_1^-)^{1/2},$$

$$\mathrm{Q}^+ = (p^{-1} + 1)\mathrm{Q}_1^+ + (p + 1)\mathrm{Q}_2^+,$$

p is the unique positive root of the equation:

$$Tr(\mathrm{Q}_2^+\mathrm{Q}_2^+)p^3 + Tr(\mathrm{Q}_1^+\mathrm{Q}_2^+)p^2 - Tr(\mathrm{Q}_1^+\mathrm{Q}_2^+)p - Tr(\mathrm{Q}_1^+\mathrm{Q}_1^+) = 0$$

(here $Tr(M)$ is the trace of $n \times n$ matrix M [2]).

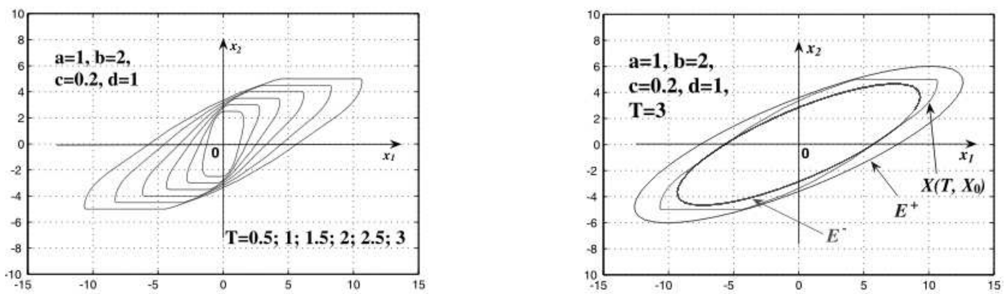
Proof. The proof of Theorem follows from Lemma, inclusions

$$\mathcal{X}_1(T, \mathcal{X}_0) \subset \mathrm{E}_1^+ = \mathcal{E}(0, \mathrm{Q}_1^+), \quad \mathcal{X}_2(T) \subset \mathrm{E}_2^+ = \mathcal{E}(0, \mathrm{Q}_2^+),$$

$$\mathcal{X}_1(T, \mathcal{X}_0) \supset \mathrm{E}_1^- = \mathcal{E}(0, \mathrm{Q}_1^-), \quad \mathcal{X}_2(T) \supset \mathrm{E}_2^- = \mathcal{E}(0, \mathrm{Q}_2^-)$$

which were proved in [6, 5] and results of [4, 1]. □

Theorem solves the problem of constructing the external and internal el-



(a) (b)

Figure 1:

lipsoidal estimate for the reachable set $\mathcal{X}(T,\mathcal{X}_0)$ of system (1) with constrains (3)-(4).

4. Example

Consider the following control system:

$$\begin{cases} dx_1(t) &= x_2(t)dt + v_1(t)dt + du_1(t), \\ dx_2(t) &= v_2(t)dt + du_2(t). \end{cases} \tag{13}$$

Here $X_0 = \{0\}$, the set $\mathcal{U}$ is generated by the ellipsoid

$$\mathcal{E}_0 = \left\{ l \in R^2 \mid l' Q_0 l \leq 1 \right\}, \quad Q_0 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}, \quad a, b \in R \ (a, b > 0),$$

and

$$v(t) \in \mathcal{E}(0, P), \quad P = \begin{pmatrix} c^2 & 0 \\ 0 & d^2 \end{pmatrix}, \quad c, d \in R \ (c, d > 0).$$

Figure 1(a) shows the dynamics of $\mathcal{X}(T)$ of the system (13). The exact reachable set $\mathcal{X}(T)$ and its external and internal ellipsoidal estimates are presented at Figure 1(b).

Acknowledgements

The research was supported by the Russian Foundation for Basic Researches (RFBR) under Project No. 06-01-00483.

References

[1] F.L. Chernousko, *State Estimation for Dynamic Systems*, Nauka, Moskow (1988).

[2] V.A. Honin, Guaranteed estimation of the state of linear systems by means of ellipsoids, In: *Evolutionary Systems in Estimation Problems* (Ed-s: A.B. Kurzhanski, T.F. Filippova), Sverdlovsk (1985), 104-123.

[3] A.B. Kurzhanski, *Control and Observation under Conditions of Uncertainty*, Nauka, Moscow (1977).

[4] A.B. Kurzhanski, I. Valyi, *Ellipsoidal Calculus for Estimation and Control*, Birkhäuser, Boston (1997).

[5] O.G. Vzdornova, Ellipsoidal techniques in state estimation problem for linear impulsive control systems, In: *Proc. of the 3-rd International IEEE Scientific Conference on Physics and Control (PhysCon 2007) (September 03-07, 2007, Potsdam, Germany)* (2007), 1-6; http://lib.physcon.ru/?item=1359

[6] O.G. Vzdornova, T.F. Filippova, External ellipsoidal estimates of the attainability sets of differential impulse systems, *J. Computer and Systems Sciences Intern.*, **45**, No. 1 (2006), 34-43.