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SEMIGROUP APPROACH OF A REPAIRABLE QUEUEING SYSTEM WITH THREE KINDS OF STATES

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Abstract: By using the Hille-Yosida Theorem, Phillips Theorem and Fattorini Theorem in functional analysis, we will prove that a repairable queueing system with three kinds of states has a unique positive time-dependent solution that satisfies probability condition.

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1. Introduction

According to Chen et al [2], the repairable queueing system with three kinds of states was described by the following system of equations:

$$\frac{dp_{1,0}(t)}{dt} = -\lambda p_{1,0}(t) + \int_0^\infty \mu_1(x) p_{1,1}(t,x)dx, \tag{1}$$

$$\frac{\partial p_{1,1}(t,x)}{\partial t} + \frac{\partial p_{1,1}(t,x)}{\partial x} = -(\alpha+\lambda+\mu_1(x))p_{1,1}(t,x), \tag{2}$$

$$\frac{\partial p_{1,n}(t,x)}{\partial t} + \frac{\partial p_{1,n}(t,x)}{\partial x} = -(\alpha+\lambda+\mu_1(x))p_{1,n}(t,x) + \lambda p_{1,n-1}(t,x), \quad n \geq 2, \tag{3}$$

$$\frac{\mathrm{d}p_{2,0}(t)}{\mathrm{d}t} = -\lambda p_{2,0}(t) + \int_0^\infty \mu_2(x) p_{2,1}(t,x)\mathrm{d}x, \tag{4}$$

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$$\frac{\partial p_{2,1}(t,x)}{\partial t}+\frac{\partial p_{2,1}(t,x)}{\partial x}=-(\gamma+\lambda+\mu_2(x))p_{2,1}(t,x), \tag{5}$$

$$\frac{\partial p_{2,n}(t,x)}{\partial t}+\frac{\partial p_{2,n}(t,x)}{\partial x}=-(\gamma+\lambda+\mu_2(x))p_{2,n}(t,x)+\lambda p_{2,n-1}(t,x),\ \ n\geq 2, \tag{6}$$

$$\frac{\partial p_{3,1}(t,x)}{\partial t}+\frac{\partial p_{3,1}(t,x)}{\partial x}=-(\lambda+\beta(x))p_{3,1}(t,x), \tag{7}$$

$$\frac{\partial p_{3,n}(t,x)}{\partial t}+\frac{\partial p_{3,n}(t,x)}{\partial x}=-(\lambda+\beta(x))p_{3,n}(t,x)+\lambda p_{3,n-1}(t,x), \quad n\geq 2, \tag{8}$$

$$p_{1,1}(t,0)=\lambda p_{1,0}(t)+\int_0^\infty \mu_1(x)p_{1,2}(t,x)\mathrm{d}x+\int_0^\infty \beta(x)p_{3,1}(t,x)\mathrm{d}x, \tag{9}$$

$$p_{1,n}(t,0)=\int_0^\infty \mu_1(x)p_{1,n+1}(t,x)\mathrm{d}x+\int_0^\infty \beta(x)p_{3,n}(t,x)\mathrm{d}x, \quad n\geq 2, \tag{10}$$

$$p_{2,1}(t,0)=\lambda p_{2,0}(t)+\int_0^\infty \mu_2(x)p_{2,2}(t,x)\mathrm{d}x+\alpha\int_0^\infty p_{1,1}(t,x)\mathrm{d}x, \tag{11}$$

$$p_{2,n}(t,0)=\int_0^\infty \mu_2(x)p_{2,n+1}(t,x)\mathrm{d}x+\alpha\int_0^\infty p_{1,n}(t,x)\mathrm{d}x, \quad n\geq 2, \tag{12}$$

$$p_{3,n}(t,0)=\gamma\int_0^\infty p_{2,n}(t,x)\mathrm{d}x, \quad n\geq 1\,, \tag{13}$$

$$p_{1,0}(0)=1, \quad p_{2,0}(0)=0, \quad p_{i,j}(0,x)=0, \quad i=1,2,3;\ j\geq 1. \tag{14}$$

Here $p_{1,0}(t)$ represents the probability that the system is empty at time t. $p_{1,n}(x,t)dx$ $(n\geq 1)$ represents the probability that at time t, the service station is normal working and there are n customers in the system with elapsed service time of the customer undergoing service lying between x and $x+dx$. $p_{2,0}(t)$ represents the probability that the system is empty at time t. $p_{2,n}(x,t)dx$ $(n\geq 1)$ represents the probability that at time t, the service station is abnormal working and there are n customers in the system with elapsed service time of the customer undergoing service lying between x and $x+dx$. $p_{3,n}(x,t)dx$ $(n\geq 1)$ represents the probability that at time t, the service station is breakdown and there are n customers in the system with elapsed service time of the customer undergoing service lying between x and $x+dx$. λ is arrival rate of customers. α is invalid rate of the service station while it turns breakdown. γ is life rate of the service station while it is abnormal. $\mu_1(x)$ represents service rate of the server when the service station is normal working. $\mu_2(x)$ represents service rate of the server when the service station is abnormal working. $\beta(x)$ represents hazard rate function when the service station is breakdown.

In Chen et al [2], the author established the above system of equations by using supplementary variable technique and discussed the time-dependent solution and steady-state solution of the above system of equations by using probability generating function and obtained expression of the probability generating functions. Roughly speaking, they obtained existence of the time-dependent solution of the above system.

In this paper, we will study the existence and uniqueness of the time-dependent solution of the above system of equations and property of the time-dependent solution. First we will convert the above system of equations into an abstract Cauchy problem by introducing suitable state space and operators, then we will prove that the operator corresponding to the above system of equations generates a positive contraction C_0-semigroup, next we will prove that the C_0-semigroup is isometric for the initial value of the system, last we will obtain that the above system of equations has a unique positive time-dependent solution which satisfies probability condition.

Choose state space $X \times Y$ as follows.

$$X \times Y = \{(p_1, p_2, p_3)| \quad p_1, p_2 \in X, p_3 \in Y,\ \|(p_1,p_2,p_3)\| = \|p_1\|_X + \|p_2\|_X + \|p_3\|_Y < \infty\},$$

$$Y = \left\{ p_3 \in L^1[0,\infty) \times L^1[0,\infty) \times \cdots \,\middle|\, \|p_3\|_Y = \sum_{k=1}^{\infty} \|p_{3,k}\|_{L^1[0,\infty)} < \infty \right\},$$

$$X = \{p_1, p_2 \in R \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots |$$

$$\|p_1\|_X = |p_{1,0}| + \sum_{k=1}^{\infty} \|p_{1,k}\|_{L^1[0,\infty)} < \infty,\ \|p_2\|_X = |p_{2,0}| + \sum_{k=1}^{\infty} \|p_{2,k}\|_{L^1[0,\infty)} < \infty\}.$$

It is obvious that $X \times Y$ is a Banach space. For simplicity, we introduce some notations as follows.

$$\Gamma_1 = \begin{pmatrix} e^{-x} & 0 & 0 & 0 & \cdots \\ \lambda e^{-x} & 0 & \mu_1(x) & 0 & \cdots \\ 0 & 0 & 0 & \mu_1(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \beta(x) & 0 & 0 & \cdots \\ 0 & \beta(x) & 0 & \cdots \\ 0 & 0 & \beta(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

$$\Gamma_3 = \begin{pmatrix} e^{-x} & 0 & 0 & 0 & \cdots \\ \lambda e^{-x} & 0 & \mu_2(x) & 0 & \cdots \\ 0 & 0 & 0 & \mu_2(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & \alpha & 0 & \cdots \\ 0 & 0 & \alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

$$\Gamma_5 = \begin{pmatrix} 0 & \gamma & 0 & 0 & \cdots \\ 0 & 0 & \gamma & 0 & \cdots \\ 0 & 0 & 0 & \gamma & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

In the following we define operators and their domains.

$$A(p_1, p_2, p_3)(x) =$$

$$\left(\begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & 0 & \cdots \\ 0 & 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} p_{1,0} \\ p_{1,1}(x) \\ p_{1,2}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & 0 & \cdots \\ 0 & 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} p_{2,0} \\ p_{2,1}(x) \\ p_{2,2}(x) \\ \vdots \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} & 0 & 0 & \cdots \\ 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & 0 & \cdots \\ 0 & 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} p_{3,1}(x) \\ p_{3,2}(x) \\ p_{3,3}(x) \\ \vdots \end{pmatrix} \right),$$

$$U(p_1, p_2, p_3)(x) =$$

$$\left(\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & -(\alpha + \lambda + \mu_1(x)) & 0 & \cdots \\ 0 & \lambda & -(\alpha + \lambda + \mu_1(x)) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} p_{1,0} \\ p_{1,1}(x) \\ p_{1,2}(x) \\ \vdots \end{pmatrix}, \right.$$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & -(\gamma + \lambda + \mu_2(x)) & 0 & \cdots \\ 0 & \lambda & -(\gamma + \lambda + \mu_2(x)) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} p_{2,0} \\ p_{2,1}(x) \\ p_{2,2}(x) \\ \vdots \end{pmatrix},$$

$$\left. \begin{pmatrix} -(\lambda + \beta(x)) & 0 & 0 & \cdots \\ \lambda & -(\lambda + \beta(x)) & 0 & \cdots \\ 0 & \lambda & -(\lambda + \beta(x)) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} p_{3,1}(x) \\ p_{3,2}(x) \\ p_{3,3}(x) \\ \vdots \end{pmatrix} \right),$$

$$E(p_1,p_2,p_3)(x)$$
$$= \left(\begin{pmatrix} \int_0^\infty \mu_1(x)p_{1,1}(x)\mathrm{d}x \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} \int_0^\infty \mu_2(x)p_{2,1}(x)\mathrm{d}x \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \right),$$

$$D(A) = \Bigg\{ (p_1,p_2,p_3) \in X \times Y \,|\, \frac{\mathrm{d}p_{1,k}(x)}{\mathrm{d}x}, \frac{\mathrm{d}p_{2,k}(x)}{\mathrm{d}x}, \frac{\mathrm{d}p_{3,k}(x)}{\mathrm{d}x} \in L^1[0,\infty), \quad k \geq 1,$$
$$p_{1,k}(x), p_{2,k}(x), p_{3,k}(x) \text{ are absolutely continuous and}$$
$$p_1(0) = \int_0^\infty \Gamma_1 p_1(x)\mathrm{d}x + \int_0^\infty \Gamma_2 p_3(x)\mathrm{d}x$$
$$p_2(0) = \int_0^\infty \Gamma_3 p_2(x)\mathrm{d}x + \int_0^\infty \Gamma_4 p_1(x)\mathrm{d}x, \quad p_3(0) = \int_0^\infty \Gamma_5 p_2(x)\mathrm{d}x \Bigg\},$$
$$D(U) = X \times Y, \quad D(E) = X \times Y.$$

Then the above system of equations (1)-(14) can be rewritten as an abstract Cauchy problem in the Banach space $X \times Y$:

$$\frac{\mathrm{d}(p_1,p_2,p_3)(t)}{\mathrm{d}t} = (A+U+E)(p_1,p_2,p_3)(t), \quad t \in (0,\infty), \tag{15}$$
$$(p_1,p_2,p_3)(0) = (1,0,0,\cdots). \tag{16}$$

Throughout this paper we assume that

$$\mu_1 = \sup_{x\in[0,\infty)} \mu_1(x) < \infty, \quad \mu_2 = \sup_{x\in[0,\infty)} \mu_2(x) < \infty, \quad \mu_3 = \sup_{x\in[0,\infty)} \beta(x) < \infty.$$

2. Main Results

Theorem 1. *$A+U+E$ generates a positive contraction C_0-semigroup $T(t)$.*

Proof. In the first step we will estimate the norm of $(\widetilde{\gamma}I - A)^{-1}$. In the second step we will prove that $D(A)$ is dense in X. Thus we will obtain that A generates a C_0-semigroup. In the third step we will prove that U and E are bounded linear operators. Thus we will deduce that $A+U+E$ generates a C_0-semigroup $T(t)$. In the fourth step we will prove that $A+U+E$ is a dispersive operator. So by combining the above steps with the Phillips Theorem we derive the desired result.

For any given $\forall\ (z_1, z_2, z_3) \in X \times Y$, we consider $(\widetilde{\gamma} I - A)(p_1, p_2, p_3) = (z_1, z_2, z_3)$. It is equivalent to

$$(\widetilde{\gamma} + \lambda) p_{1,0} = z_{1,0}, \tag{17}$$

$$(\widetilde{\gamma} + \lambda) p_{2,0} = z_{2,0}, \tag{18}$$

$$\frac{\mathrm{d}p_{1,n}}{\mathrm{d}x} = -\widetilde{\gamma} p_{1,n} + z_{1,n}, \quad n \geq 1, \tag{19}$$

$$\frac{\mathrm{d}p_{2,n}}{\mathrm{d}x} = -\widetilde{\gamma} p_{2,n} + z_{2,n}, \quad n \geq 1, \tag{20}$$

$$\frac{\mathrm{d}p_{3,n}}{\mathrm{d}x} = -\widetilde{\gamma} p_{3,n} + z_{3,n}, \quad n \geq 1, \tag{21}$$

$$p_{1,1}(0) = \lambda p_{1,0}(t) + \int_0^\infty \mu_1(x) p_{1,2}(x) \mathrm{d}x + \int_0^\infty \beta(x) p_{3,1}(x) \mathrm{d}x, \tag{22}$$

$$p_{1,n}(0) = \int_0^\infty \mu_1(x) p_{1\,n+1}(x) \mathrm{d}x + \int_0^\infty \beta(x) p_{3,n}(x) \mathrm{d}x, \quad n \geq 2, \tag{23}$$

$$p_{2,1}(0) = \lambda p_{2,0}(t) + \int_0^\infty \mu_2(x) p_{2,2}(x) \mathrm{d}x + \alpha \int_0^\infty p_{1,1}(t,x) \mathrm{d}x, \tag{24}$$

$$p_{2,n}(0) = \int_0^\infty \mu_2(x) p_{2,n+1}(x) \mathrm{d}x + \alpha \int_0^\infty p_{1,n}(x) \mathrm{d}x, \quad n \geq 2, \tag{25}$$

$$p_{3,n}(0) = \gamma \int_0^\infty p_{2,n}(x) \mathrm{d}x, \quad n \geq 1. \tag{26}$$

By solving (17)-(21) we have

$$p_{1,0} = \frac{1}{\widetilde{\gamma} + \lambda} z_{1,0}, \tag{27}$$

$$p_{2,0} = \frac{1}{\widetilde{\gamma} + \lambda} z_{2,0}, \tag{28}$$

$$p_{1,n}(x) = a_n e^{-\widetilde{\gamma} x} + e^{-\widetilde{\gamma} x} \int_0^x z_{1,n}(\tau) e^{\widetilde{\gamma} \tau} \mathrm{d}\tau, \quad n \geq 1, \tag{29}$$

$$p_{2,n}(x) = b_n e^{-\widetilde{\gamma} x} + e^{-\widetilde{\gamma} x} \int_0^x z_{2,n}(\tau) e^{\widetilde{\gamma} \tau} \mathrm{d}\tau, \quad n \geq 1, \tag{30}$$

$$p_{3,n}(x) = c_n e^{-\widetilde{\gamma} x} + e^{-\widetilde{\gamma} x} \int_0^x z_{3,n}(\tau) e^{\widetilde{\gamma} \tau} \mathrm{d}\tau, \quad n \geq 1. \tag{31}$$

By combining (22)-(26) with (27)-(31) and the Fubini Theorem we deduce

$$\begin{aligned} a_1 = & \frac{\lambda}{\widetilde{\gamma} + \lambda} z_{1,0} + a_2 \int_0^\infty \mu_1(x) e^{-\widetilde{\gamma} x} \mathrm{d}x + \int_0^\infty \mu_1(x) e^{-\widetilde{\gamma} x} \int_0^x z_{1,2}(\tau) e^{\widetilde{\gamma} \tau} \mathrm{d}\tau \mathrm{d}x \\ & + c_1 \int_0^\infty \beta(x) e^{-\widetilde{\gamma} x} \mathrm{d}x + \int_0^\infty \beta(x) e^{-\widetilde{\gamma} x} \int_0^x z_{3,1}(\tau) e^{\widetilde{\gamma} \tau} \mathrm{d}\tau \mathrm{d}x, \end{aligned} \tag{32}$$

$$\begin{aligned} a_n =& a_{n+1}\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\int_0^x z_{1,n+1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \\ &+ c_n\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\int_0^x z_{3,n}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x, \quad n\geq 2, \quad (33) \\ b_1 =& \frac{\lambda}{\widetilde{\gamma}+\lambda}z_{2,0} + b_2\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,2}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \\ &+ \frac{\alpha}{\widetilde{\gamma}}a_1 + \alpha\int_0^\infty e^{-\widetilde{\gamma}x}\int_0^x z_{1,1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x, \quad (34) \\ b_n =& b_{n+1}\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,n+1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \\ &+ \frac{\alpha}{\widetilde{\gamma}}a_n + \alpha\int_0^\infty e^{-\widetilde{\gamma}x}\int_0^x z_{1,n}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x, \quad n\geq 2, \quad (35) \\ c_n =& \frac{\gamma}{\widetilde{\gamma}}b_n + \gamma\int_0^\infty e^{-\widetilde{\gamma}x}\int_0^x z_{2,n}(\tau)e^{\widetilde{\gamma}\tau}d\tau dx = \frac{\gamma}{\widetilde{\gamma}}b_n + \frac{\gamma}{\widetilde{\gamma}}\int_0^\infty z_{2,n}(\tau)\mathrm{d}\tau, \quad n\geq 1. \\ & \quad (36) \end{aligned}$$

From (36) and (32) it follows that

$$\begin{aligned} a_1 =& \frac{\lambda}{\widetilde{\gamma}+\lambda}z_{1,0} + a_2\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\int_0^x z_{1,2}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \\ &+ \frac{\gamma}{\widetilde{\gamma}}b_1\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \frac{\gamma}{\widetilde{\gamma}}\int_0^\infty z_{2,1}(\tau)\mathrm{d}\tau\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \\ &+ \int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\int_0^x z_{3,1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x. \quad (37) \end{aligned}$$

By (34), (37) and the Fubini Theorem we know

$$\begin{aligned} b_1 =& \frac{\lambda}{\widetilde{\gamma}+\lambda}z_{2,0} + \frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}z_{1,0} + b_2\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \\ &+ \int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,2}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \\ &+ \frac{\alpha}{\widetilde{\gamma}}a_2\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \frac{\alpha}{\widetilde{\gamma}}\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\int_0^x z_{1,2}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \\ &+ \frac{\alpha\gamma}{\widetilde{\gamma}^2}b_1\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \frac{\alpha\gamma}{\widetilde{\gamma}^2}\int_0^\infty z_{2,1}(\tau)\mathrm{d}\tau\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \\ &+ \frac{\alpha}{\widetilde{\gamma}}\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\int_0^x z_{3,1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x + \frac{\alpha}{\widetilde{\gamma}}\int_0^\infty z_{1,1}(\tau)\mathrm{d}\tau. \quad (38) \end{aligned}$$

From (35) and the Fubini Theorem we have

$$\begin{aligned}\frac{\alpha}{\widetilde{\gamma}}a_2 =& b_2 - b_3\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\mathrm{d}x - \int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,3}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\\ &- \frac{\alpha}{\widetilde{\gamma}}\int_0^\infty z_{1,2}(\tau)\mathrm{d}\tau. \end{aligned}\tag{39}$$

By inserting (39) into (38) we derive

$$\begin{aligned}b_1 =& \frac{\lambda}{\widetilde{\gamma}+\lambda}z_{2,0} + \frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}z_{1,0} + b_2\int_0^\infty [\mu_1(x)+\mu_2(x)]e^{-\widetilde{\gamma}x}\mathrm{d}x\\ &+ \int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,2}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\\ &- b_3\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \times \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x\\ &- \int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,3}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \times \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x\\ &- \frac{\alpha}{\widetilde{\gamma}}\int_0^\infty z_{1,2}(\tau)\mathrm{d}\tau \times \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \frac{\alpha}{\widetilde{\gamma}}\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\int_0^x z_{1,2}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\\ &+ \frac{\alpha\gamma}{\widetilde{\gamma}^2}b_1\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \frac{\alpha\gamma}{\widetilde{\gamma}^2}\int_0^\infty z_{2,1}(\tau)\mathrm{d}\tau\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x\\ &+ \frac{\alpha}{\widetilde{\gamma}}\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\int_0^x z_{3,1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x + \frac{\alpha}{\widetilde{\gamma}}\int_0^\infty z_{1,1}(\tau)\mathrm{d}\tau. \end{aligned}\tag{40}$$

Through substituting (36) into (33) we calculate

$$\begin{aligned}a_n =& a_{n+1}\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\int_0^x z_{1,n+1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\\ &+ \frac{\gamma}{\widetilde{\gamma}}b_n\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \frac{\gamma}{\widetilde{\gamma}}\int_0^\infty z_{2,n}(\tau)\mathrm{d}\tau\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x\\ &+ \int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\int_0^x z_{3,n}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x, \quad n \geq 2. \end{aligned}\tag{41}$$

By inserting (41) into (35) and using the Fubini Theorem we obtain

$$\begin{aligned}b_n =& b_{n+1}\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,n+1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\\ &+ \frac{\alpha}{\widetilde{\gamma}}a_{n+1}\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \frac{\alpha}{\widetilde{\gamma}}\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\int_0^x z_{1,n+1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\\ &+ \frac{\alpha\gamma}{\widetilde{\gamma}^2}b_n\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \frac{\alpha\gamma}{\widetilde{\gamma}^2}\int_0^\infty z_{2,n}(\tau)\mathrm{d}\tau\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \end{aligned}$$

$$+\frac{\alpha}{\tilde{\tilde{\gamma}}}\int_0^\infty \beta(x)e^{-\tilde{\gamma}x}\int_0^x z_{3,n}(\tau)e^{\tilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x+\frac{\alpha}{\tilde{\tilde{\gamma}}}\int_0^\infty z_{1,n}(\tau)\mathrm{d}\tau,\quad n\geq 2. \quad (42)$$

Through rearranging terms in (35) and using the Fubini Theorem it follows that

$$\frac{\alpha}{\tilde{\tilde{\gamma}}}a_{n+1}=b_{n+1}-b_{n+2}\int_0^\infty \mu_2(x)e^{-\tilde{\gamma}x}\mathrm{d}x-\int_0^\infty \mu_2(x)e^{-\tilde{\gamma}x}\int_0^x z_{2,n+2}(\tau)e^{\tilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x$$
$$-\frac{\alpha}{\tilde{\tilde{\gamma}}}\int_0^\infty z_{1,n+1}(\tau)\mathrm{d}\tau,\quad n\geq 2. \quad (43)$$

By inserting (43) into (42) we know

$$\begin{aligned}
b_n =&b_{n+1}\int_0^\infty[\mu_1(x)+\mu_2(x)]e^{-\tilde{\gamma}x}\mathrm{d}x+\int_0^\infty \mu_2(x)e^{-\tilde{\gamma}x}\int_0^x z_{2,n+1}(\tau)e^{\tilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\\
&-b_{n+2}\int_0^\infty \mu_2(x)e^{-\tilde{\gamma}x}\mathrm{d}x\times\int_0^\infty \mu_1(x)e^{-\tilde{\gamma}x}\mathrm{d}x\\
&-\int_0^\infty \mu_2(x)e^{-\tilde{\gamma}x}\int_0^x z_{2,n+2}(\tau)e^{\tilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\times\int_0^\infty \mu_1(x)e^{-\tilde{\gamma}x}\mathrm{d}x\\
&-\frac{\alpha}{\tilde{\tilde{\gamma}}}\int_0^\infty z_{1,n+1}(\tau)\mathrm{d}\tau\times\int_0^\infty \mu_1(x)e^{-\tilde{\gamma}x}\mathrm{d}x\\
&+\frac{\alpha}{\tilde{\tilde{\gamma}}}\int_0^\infty \mu_1(x)e^{-\tilde{\gamma}x}\int_0^x z_{1,n+1}(\tau)e^{\tilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\\
&+\frac{\alpha\gamma}{\tilde{\gamma}^2}b_n\int_0^\infty \beta(x)e^{-\tilde{\gamma}x}\mathrm{d}x+\frac{\alpha\gamma}{\tilde{\gamma}^2}\int_0^\infty z_{2,n}(\tau)\mathrm{d}\tau\int_0^\infty \beta(x)e^{-\tilde{\gamma}x}\mathrm{d}x\\
&+\frac{\alpha}{\tilde{\tilde{\gamma}}}\int_0^\infty \beta(x)e^{-\tilde{\gamma}x}\int_0^x z_{3,n}(\tau)e^{\tilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x+\frac{\alpha}{\tilde{\tilde{\gamma}}}\int_0^\infty z_{1,n}(\tau)\mathrm{d}\tau,\quad n\geq 2. \quad (44)
\end{aligned}$$

If we define

$$F=\begin{pmatrix} a & b & c & 0 & 0 & 0 & \cdots\\ 0 & a & b & c & 0 & 0 & \cdots\\ 0 & 0 & a & b & c & 0 & \cdots\\ 0 & 0 & 0 & a & b & c & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},\qquad \overrightarrow{b}=\begin{pmatrix} b_1\\ b_2\\ b_3\\ b_4\\ \vdots\end{pmatrix},$$

then (40) and (44) are equivalent to

$$F\overrightarrow{b}=\left(\frac{\lambda}{\tilde{\gamma}+\lambda}z_{2,0}+\frac{\alpha\lambda}{\tilde{\gamma}(\tilde{\gamma}+\lambda)}z_{1,0}+d_1,d_2,d_3,\cdots,d_m,\cdots\right)^T. \quad (45)$$

Here

$$\begin{aligned}a =&1-\frac{\alpha\gamma}{\widetilde{\gamma}^2}\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x,\\ b =&-\int_0^\infty [\mu_1(x)+\mu_2(x)]e^{-\widetilde{\gamma}x}\mathrm{d}x,\\ c =&\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x\times\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}d_m =& \frac{\alpha}{\widetilde{\widetilde{\gamma}}}\int_0^\infty z_{1,m}(\tau)\mathrm{d}\tau-\frac{\alpha}{\widetilde{\widetilde{\gamma}}}\int_0^\infty z_{1,m+1}(\tau)\mathrm{d}\tau\times\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x\\ &+\frac{\alpha}{\widetilde{\widetilde{\gamma}}}\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\int_0^x z_{1,m+1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x+\frac{\alpha\gamma}{\widetilde{\widetilde{\gamma}}^2}\int_0^\infty z_{2,m}(\tau)\mathrm{d}\tau\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x\\ &+\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,m+1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x-\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,m+2}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\\ &\times\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x+\frac{\alpha}{\widetilde{\widetilde{\gamma}}}\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\int_0^x z_{3,m}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x,\quad m\geq 1.\end{aligned}$$

It is not difficult to calculate

$$F^{-1}=\begin{pmatrix}\frac{1}{a} & \frac{-b}{a^2} & \frac{b^2}{a^3}+\frac{-c}{a^2} & \frac{-b^3}{a^4}+\frac{2bc}{a^3} & \frac{b^4}{a^5}+\frac{-3b^2c}{a^4}+\frac{c^2}{a^3} & \cdots & \cdots & \sum_{k=0}^{[\frac{n-1}{2}]}\frac{(-1)^{n-1-k}C_{n-1-k}^k b^{n-1-2k}c^k}{a^{n-k}} & \cdots\\ 0 & \frac{1}{a} & \frac{-b}{a^2} & \frac{b^2}{a^3}+\frac{-c}{a^2} & \frac{-b^3}{a^4}+\frac{2bc}{a^3} & \cdots & \cdots & \sum_{k=0}^{[\frac{n-2}{2}]}\frac{(-1)^{n-2-k}C_{n-2-k}^k b^{n-2-2k}c^k}{a^{n-1-k}} & \cdots\\ 0 & 0 & \frac{1}{a} & \frac{-b}{a^2} & \frac{b^2}{a^3}+\frac{-c}{a^2} & \cdots & \cdots & \sum_{k=0}^{[\frac{n-3}{2}]}\frac{(-1)^{n-3-k}C_{n-3-k}^k b^{n-3-2k}c^k}{a^{n-2-k}} & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{pmatrix}.$$

From which together with (45) we have

$$b_1=\Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}z_{2,0}+\frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}z_{1,0}\Big)\frac{1}{a}+\sum_{m=1}^{\infty}\sum_{k=0}^{\left[\frac{m-1}{2}\right]}\frac{(-1)^{m-1-k}C_{m-1-k}^{k}b^{m-1-2k}c^{k}}{a^{m-k}}d_m$$
$$=\Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}z_{2,0}+\frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}z_{1,0}\Big)\frac{1}{a}+\sum_{m=0}^{\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]}\frac{(-1)^{m-k}C_{m-k}^{k}b^{m-2k}c^{k}}{a^{m+1-k}}d_{m+1}, \quad (46)$$

$$b_n=\sum_{m=0}^{\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]}\frac{(-1)^{m-k}C_{m-k}^{k}b^{m-2k}c^{k}}{a^{m+1-k}}d_{m+n}, \quad n\geq 2. \quad (47)$$

By combining (46) and (47) with the Fubini Theorem we estimate (without loss of generality, assume $\widetilde{\gamma}>\max\{0,2(\mu_1+\mu_2)+\alpha,\sqrt{\gamma\mu_3}\}$)

$$\sum_{n=1}^{\infty}|b_n|=|b_1|+\sum_{n=2}^{\infty}|b_n|$$
$$\leq\Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{2,0}|+\frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}|z_{1,0}|\Big)\frac{1}{|a|}+\sum_{m=0}^{\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]}\frac{C_{m-k}^{k}|b|^{m-2k}c^{k}}{a^{m+1-k}}|d_{m+1}|$$
$$+\sum_{n=2}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]}\frac{C_{m-k}^{k}|b|^{m-2k}c^{k}}{a^{m+1-k}}|d_{m+n}|$$
$$=\Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{2,0}|+\frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}|z_{1,0}|\Big)\frac{1}{|a|}+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]}\frac{C_{m-k}^{k}b^{m-2k}c^{k}}{a^{m+1-k}}|d_{m+n}|$$
$$\leq\Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{2,0}|+\frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}|z_{1,0}|\Big)\frac{1}{1-\frac{\alpha\gamma}{\widetilde{\gamma}^2}\int_0^\infty\beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x}+$$
$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]}\frac{C_{m-k}^{k}\big(\int_0^\infty[\mu_1(x)+\mu_2(x)]e^{-\widetilde{\gamma}x}\mathrm{d}x\big)^{m-2k}\big(\int_0^\infty\mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x\int_0^\infty\mu_2(x)e^{-\widetilde{\gamma}x}\mathrm{d}x\big)^{k}}{\big(1-\frac{\alpha\gamma}{\widetilde{\gamma}^2}\int_0^\infty\beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x\big)^{m+1-k}}$$
$$\Big|\frac{\alpha}{\widetilde{\gamma}}\int_0^\infty z_{1,m+n}(\tau)\mathrm{d}\tau-\frac{\alpha}{\widetilde{\gamma}}\int_0^\infty z_{1,m+1+n}(\tau)\mathrm{d}\tau\times\int_0^\infty\mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x$$
$$+\frac{\alpha}{\widetilde{\gamma}}\int_0^\infty\mu_1(x)e^{-\widetilde{\gamma}x}\int_0^x z_{1,m+1+n}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x+\frac{\alpha\gamma}{\widetilde{\gamma}^2}\int_0^\infty z_{2,m+n}(\tau)\mathrm{d}\tau\int_0^\infty\beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x$$
$$+\int_0^\infty\mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,m+1+n}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x-\int_0^\infty\mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,m+2+n}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x$$
$$\times\int_0^\infty\mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x+\frac{\alpha}{\widetilde{\gamma}}\int_0^\infty\beta(x)e^{-\widetilde{\gamma}x}\int_0^x z_{3,m+n}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x\Big|$$

$$\le \Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}\big|z_{2,0}\big| + \frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}\big|z_{1,0}\big|\Big)\frac{1}{1-\frac{\alpha\gamma}{\widetilde{\gamma}^2}\int_0^\infty \mu_3 e^{-\widetilde{\gamma}x}\mathrm{d}x}$$

$$+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{[\frac{m}{2}]}\frac{C_{m-k}^{k}\big(\int_0^\infty(\mu_1+\mu_2)e^{-\widetilde{\gamma}x}\mathrm{d}x\big)^{m-2k}\big(\int_0^\infty \mu_1 e^{-\widetilde{\gamma}x}\mathrm{d}x\times\int_0^\infty \mu_2 e^{-\widetilde{\gamma}x}\mathrm{d}x\big)^k}{\big(1-\frac{\alpha\gamma}{\widetilde{\gamma}^2}\int_0^\infty \mu_3 e^{-\widetilde{\gamma}x}\mathrm{d}x\big)^{m+1-k}}$$

$$\times\Bigg\{\frac{\alpha}{\widetilde{\gamma}}\|z_{1,m+n}\|_{L^1[0,\infty)} + \frac{\alpha\mu_1}{\widetilde{\gamma}^2}\|z_{1,m+1+n}\|_{L^1[0,\infty)} + \frac{\alpha\mu_1}{\widetilde{\gamma}^2}\|z_{1,m+1+n}\|_{L^1[0,\infty)}$$

$$+\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\|z_{2,m+n}\|_{L^1[0,\infty)} + \frac{\mu_2}{\widetilde{\gamma}}\|z_{2,m+1+n}\|_{L^1[0,\infty)} + \frac{\mu_1\mu_2}{\widetilde{\gamma}^2}\|z_{2,m+2+n}\|_{L^1[0,\infty)}$$

$$+\frac{\alpha\mu_3}{\widetilde{\gamma}^2}\|z_{3,m+n}\|_{L^1[0,\infty)}\Bigg\}$$

$$\le \left(\frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{2,0}| + \frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}|z_{1,0}|\right)\frac{1}{1-\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}}$$

$$+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{[\frac{m}{2}]}\frac{C_{m-k}^{k}\big(\frac{\mu_1+\mu_2}{\widetilde{\gamma}}\big)^{m-2k}\big(\frac{\mu_1\mu_2}{\widetilde{\gamma}^2}\big)^k}{\big(1-\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\big)^{m+1-k}}$$

$$\times\Bigg\{\frac{\alpha}{\widetilde{\gamma}}\|z_{1,m+n}\|_{L^1[0,\infty)} + \frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\|z_{2,m+n}\|_{L^1[0,\infty)} + \frac{\alpha\mu_3}{\widetilde{\gamma}^2}\|z_{3,m+n}\|_{L^1[0,\infty)}$$

$$+\frac{2\alpha\mu_1}{\widetilde{\gamma}^2}\|z_{1,m+n+1}\|_{L^1[0,\infty)} + \frac{\mu_2}{\widetilde{\gamma}}\|z_{2,m+n+1}\|_{L^1[0,\infty)} + \frac{\mu_1\mu_2}{\widetilde{\gamma}^2}\|z_{2,m+n+2}\|_{L^1[0,\infty)}\Bigg\}$$

$$\le \Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}\big|z_{2,0}\big| + \frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}\big|z_{1,0}\big|\Big)\frac{\widetilde{\gamma}^3}{\widetilde{\gamma}^3-\alpha\gamma\mu_3} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{[\frac{m}{2}]}\frac{2^{m-k}\big(\frac{\mu_1+\mu_2}{\widetilde{\gamma}}\big)^{m-2k}\big(\frac{\mu_1\mu_2}{\widetilde{\gamma}^2}\big)^k}{\big(1-\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\big)^{m+1-k}}$$

$$\times\Bigg\{\frac{\alpha}{\widetilde{\gamma}}\|z_{1,m+n}\|_{L^1[0,\infty)} + \frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\|z_{2,m+n}\|_{L^1[0,\infty)} + \frac{\alpha\mu_3}{\widetilde{\gamma}^2}\|z_{3,m+n}\|_{L^1[0,\infty)}$$

$$+\frac{2\alpha\mu_1}{\widetilde{\gamma}^2}\|z_{1,m+n+1}\|_{L^1[0,\infty)} + \frac{\mu_2}{\widetilde{\gamma}}\|z_{2,m+n+1}\|_{L^1[0,\infty)} + \frac{\mu_1\mu_2}{\widetilde{\gamma}^2}\|z_{2,m+n+2}\|_{L^1[0,\infty)}\Bigg\}$$

$$\le \Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}\big|z_{2,0}\big| + \frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}\big|z_{1,0}\big|\Big)\frac{\widetilde{\gamma}^3}{\widetilde{\gamma}^3-\alpha\gamma\mu_3} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{[\frac{m}{2}]}\frac{2^{m-k}\big(\frac{\mu_1+\mu_2}{\widetilde{\gamma}}\big)^{m-2k}\big(\frac{\mu_1+\mu_2}{2\widetilde{\gamma}}\big)^{2k}}{\big(1-\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\big)^{m+1-k}}$$

$$\times\Bigg\{\frac{\alpha}{\widetilde{\gamma}}\|z_{1,m+n}\|_{L^1[0,\infty)} + \frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\|z_{2,m+n}\|_{L^1[0,\infty)} + \frac{\alpha\mu_3}{\widetilde{\gamma}^2}\|z_{3,m+n}\|_{L^1[0,\infty)}$$

$$+\frac{2\alpha\mu_1}{\widetilde{\gamma}^2}\|z_{1,m+n+1}\|_{L^1[0,\infty)}+\frac{\mu_2}{\widetilde{\gamma}}\|z_{2,m+n+1}\|_{L^1[0,\infty)}+\frac{\mu_1\mu_2}{\widetilde{\gamma}^2}\|z_{2,m+n+2}\|_{L^1[0,\infty)}\Bigg\}$$

$$\leq\Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{2,0}|+\frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}|z_{1,0}|\Big)\frac{\widetilde{\gamma}^3}{\widetilde{\gamma}^3-\alpha\gamma\mu_3}+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{2(\mu_1+\mu_2)}{\widetilde{\gamma}}\right)^m}{\left(1-\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\right)^{m+1}}\sum_{k=0}^{\left[\frac{m}{2}\right]}\frac{8^{-k}}{\left(1-\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\right)^{-k}}$$

$$\times\Bigg\{\frac{\alpha}{\widetilde{\gamma}}\|z_{1,m+n}\|_{L^1[0,\infty)}+\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\|z_{2,m+n}\|_{L^1[0,\infty)}+\frac{\alpha\mu_3}{\widetilde{\gamma}^2}\|z_{3,m+n}\|_{L^1[0,\infty)}$$

$$+\frac{2\alpha\mu_1}{\widetilde{\gamma}^2}\|z_{1,m+n+1}\|_{L^1[0,\infty)}+\frac{\mu_2}{\widetilde{\gamma}}\|z_{2,m+n+1}\|_{L^1[0,\infty)}+\frac{\mu_1\mu_2}{\widetilde{\gamma}^2}\|z_{2,m+n+2}\|_{L^1[0,\infty)}\Bigg\}$$

$$=\Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{2,0}|+\frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}|z_{1,0}|\Big)\frac{\widetilde{\gamma}^3}{\widetilde{\gamma}^3-\alpha\gamma\mu_3}$$

$$+\frac{\widetilde{\gamma}^3}{\widetilde{\gamma}^3-\alpha\gamma\mu_3}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left[\frac{2(\mu_1+\mu_2)\widetilde{\gamma}^2}{\widetilde{\gamma}^3-\alpha\gamma\mu_3}\right]^m\sum_{k=0}^{\left[\frac{m}{2}\right]}\left[\frac{\widetilde{\gamma}^3-\alpha\gamma\mu_3}{8\widetilde{\gamma}^3}\right]^k$$

$$\times\Bigg\{\frac{\alpha}{\widetilde{\gamma}}\|z_{1,m+n}\|_{L^1[0,\infty)}+\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\|z_{2,m+n}\|_{L^1[0,\infty)}+\frac{\alpha\mu_3}{\widetilde{\gamma}^2}\|z_{3,m+n}\|_{L^1[0,\infty)}$$

$$+\frac{2\alpha\mu_1}{\widetilde{\gamma}^2}\|z_{1,m+n+1}\|_{L^1[0,\infty)}+\frac{\mu_2}{\widetilde{\gamma}}\|z_{2,m+n+1}\|_{L^1[0,\infty)}+\frac{\mu_1\mu_2}{\widetilde{\gamma}^2}\|z_{2,m+n+2}\|_{L^1[0,\infty)}\Bigg\}$$

$$\leq\Big(\frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{2,0}|+\frac{\alpha\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}|z_{1,0}|\Big)\frac{\widetilde{\gamma}^3}{\widetilde{\gamma}^3-\alpha\gamma\mu_3}$$

$$+\frac{\widetilde{\gamma}^3}{\widetilde{\gamma}^3-\alpha\gamma\mu_3}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left[\frac{2(\mu_1+\mu_2)\widetilde{\gamma}^2}{\widetilde{\gamma}^3-\alpha\gamma\mu_3}\right]^m\frac{1}{1-\frac{\widetilde{\gamma}^3-\alpha\gamma\mu_3}{8\widetilde{\gamma}^3}}$$

$$\times\Bigg\{\frac{\alpha}{\widetilde{\gamma}}\|z_{1,m+n}\|_{L^1[0,\infty)}+\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\|z_{2,m+n}\|_{L^1[0,\infty)}+\frac{\alpha\mu_3}{\widetilde{\gamma}^2}\|z_{3,m+n}\|_{L^1[0,\infty)}$$

$$+\frac{2\alpha\mu_1}{\widetilde{\gamma}^2}\|z_{1,m+n+1}\|_{L^1[0,\infty)}+\frac{\mu_2}{\widetilde{\gamma}}\|z_{2,m+n+1}\|_{L^1[0,\infty)}+\frac{\mu_1\mu_2}{\widetilde{\gamma}^2}\|z_{2,m+n+2}\|_{L^1[0,\infty)}\Bigg\}$$

$$\leq\frac{\lambda\widetilde{\gamma}^3}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{2,0}|+\frac{\alpha\lambda\widetilde{\gamma}^2}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{1,0}|$$

$$+\frac{\widetilde{\gamma}^3}{\widetilde{\gamma}^3-\alpha\gamma\mu_3}\frac{1}{1-\frac{2(\mu_1+\mu_2)\widetilde{\gamma}^2}{\widetilde{\gamma}^3-\alpha\gamma\mu_3}}\frac{1}{1-\frac{\widetilde{\gamma}^3-\alpha\gamma\mu_3}{8\widetilde{\gamma}^3}}$$

$$\times\Bigg\{\frac{\alpha}{\widetilde{\gamma}}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)}+\frac{\alpha\gamma\mu_3}{\widetilde{\gamma}^3}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}+\frac{\alpha\mu_3}{\widetilde{\gamma}^2}\sum_{n=1}^{\infty}\|z_{3,n}\|_{L^1[0,\infty)}$$

$$+\frac{2\alpha\mu_1}{\widetilde{\gamma}^2}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)}+\frac{\mu_2}{\widetilde{\gamma}}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}+\frac{\mu_1\mu_2}{\widetilde{\gamma}^2}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}\Bigg\}$$

$$
\begin{aligned}
&= \frac{\lambda\widetilde{\gamma}^3}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{2,0}| + \frac{\alpha\lambda\widetilde{\gamma}^2}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{1,0}| \\
&+ \frac{\widetilde{\gamma}^3}{\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3}\frac{8\widetilde{\gamma}^3}{7\widetilde{\gamma}^3+\alpha\gamma\mu_3}\Bigg\{\frac{\alpha\widetilde{\gamma}+2\alpha\mu_1}{\widetilde{\gamma}^2}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)} \\
&+\frac{\alpha\gamma\mu_3+\mu_1\mu_2\widetilde{\gamma}+\mu_2\widetilde{\gamma}^2}{\widetilde{\gamma}^3}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)} + \frac{\alpha\mu_3}{\widetilde{\gamma}^2}\|z_3\|_Y\Bigg\} \\
&= \frac{\lambda\widetilde{\gamma}^3}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{2,0}| + \frac{\alpha\lambda\widetilde{\gamma}^2}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{1,0}| \\
&+ \frac{8\widetilde{\gamma}^4(\alpha\widetilde{\gamma}+2\alpha\mu_1)}{(\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3)(7\widetilde{\gamma}^3+\alpha\gamma\mu_3)}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)} \\
&+ \frac{8\widetilde{\gamma}^3(\mu_2\widetilde{\gamma}^2+\mu_1\mu_2\widetilde{\gamma}+\alpha\gamma\mu_3)}{(\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3)(7\widetilde{\gamma}^3+\alpha\gamma\mu_3)}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)} \\
&+ \frac{8\widetilde{\gamma}^4\alpha\mu_3}{(\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3)(7\widetilde{\gamma}^3+\alpha\gamma\mu_3)}\|z_3\|_Y. \quad (48)
\end{aligned}
$$

In (48) we used the following inequalities.

$2^{m-k} = (1+1)^{m-k} = C^0_{m-k} + C^1_{m-k} + \cdots + C^{m-k}_{m-k} \geq C^k_{m-k}$, for $0 \leq k \leq \left[\frac{m}{2}\right]$,

$(\frac{\mu_1+\mu_2}{2})^2 = \frac{\mu_1^2+\mu_2^2}{4} + \frac{\mu_1\mu_2}{2} \geq \frac{2\mu_1\mu_2}{4} + \frac{\mu_1\mu_2}{2} = \mu_1\mu_2\,,$

$\frac{2(\mu_1+\mu_2)\widetilde{\gamma}^2}{\widetilde{\gamma}^3-\alpha\gamma\mu_3} < 1$, as $\widetilde{\gamma}^3 - 2(\mu_1+\mu_2)\widetilde{\gamma}^2 > \alpha\gamma\mu_3\,,$

$\frac{\widetilde{\gamma}^3-\alpha\gamma\mu_3}{8\widetilde{\gamma}^3} < 1$, for $\widetilde{\gamma} > 0$.

By combining (32) and (35) with (36) and the Fubini Theorem we have

$$
\begin{aligned}
a_1 &= \frac{\lambda}{\widetilde{\gamma}+\lambda}z_{1,0} + \frac{\gamma}{\widetilde{\gamma}}\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \times b_1 + \frac{\widetilde{\gamma}}{\alpha}\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \times b_2 \\
&\quad - \frac{\widetilde{\gamma}}{\alpha}\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \times \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \times b_3 \\
&+ \frac{\gamma}{\widetilde{\gamma}}\int_0^\infty z_{2,1}(\tau)\mathrm{d}\tau \times \int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\int_0^x z_{3,1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \\
&\quad - \frac{\widetilde{\gamma}}{\alpha}\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,3}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \times \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \\
&- \int_0^\infty z_{1,2}(\tau)\mathrm{d}\tau\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\int_0^x z_{1,2}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x. \quad (49)
\end{aligned}
$$

By using (33), (35) and (36) we deduce

$$\begin{aligned} a_n =&\frac{\gamma}{\widetilde{\gamma}}\int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \times b_n + \frac{\widetilde{\gamma}}{\alpha}\int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \times b_{n+1} \\ &- \frac{\widetilde{\gamma}}{\alpha}\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \times \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \times b_{n+2} \\ &+ \frac{\gamma}{\widetilde{\gamma}}\int_0^\infty z_{2,n}(\tau)\mathrm{d}\tau \times \int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\mathrm{d}x + \int_0^\infty \beta(x)e^{-\widetilde{\gamma}x}\int_0^x z_{3,n}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \\ &- \frac{\widetilde{\gamma}}{\alpha}\int_0^\infty \mu_2(x)e^{-\widetilde{\gamma}x}\int_0^x z_{2,n+2}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x \times \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \\ &- \int_0^\infty z_{1,n+1}(\tau)\mathrm{d}\tau \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\mathrm{d}x \\ &+ \int_0^\infty \mu_1(x)e^{-\widetilde{\gamma}x}\int_0^x z_{1,n+1}(\tau)e^{\widetilde{\gamma}\tau}\mathrm{d}\tau\mathrm{d}x, \quad n \geq 2. \end{aligned} \tag{50}$$

From (49), (50) and the Fubini Theorem it follows that

$$\begin{aligned} &\sum_{n=1}^\infty |a_n| = |a_1| + \sum_{n=2}^\infty |a_n| \\ &\leq \frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{1,0}| + \frac{\gamma\mu_3}{\widetilde{\gamma}^2}\sum_{n=1}^\infty |b_n| + \frac{\mu_1}{\alpha}\sum_{n=1}^\infty |b_{n+1}| + \frac{\mu_1\mu_2}{\alpha\widetilde{\gamma}}\sum_{n=1}^\infty |b_{n+2}| + \frac{\gamma\mu_3}{\widetilde{\gamma}^2}\sum_{n=1}^\infty \|z_{2,n}\|_{L^1[0,\infty)} \\ &\quad + \frac{\mu_3}{\widetilde{\gamma}}\sum_{n=1}^\infty \|z_{3,n}\|_{L^1[0,\infty)} + \frac{\mu_1\mu_2}{\alpha\widetilde{\gamma}}\sum_{n=1}^\infty \|z_{2,n+2}\|_{L^1[0,\infty)} + \frac{2\mu_1}{\widetilde{\gamma}}\sum_{n=1}^\infty \|z_{1,n+1}\|_{L^1[0,\infty)} \\ &\qquad \leq \frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{1,0}| + \left\{\frac{\gamma\mu_3}{\widetilde{\gamma}^2} + \frac{\mu_1}{\alpha} + \frac{\mu_1\mu_2}{\alpha\widetilde{\gamma}}\right\}\sum_{n=1}^\infty |b_n| + \frac{2\mu_1}{\widetilde{\gamma}}\sum_{n=1}^\infty \|z_{1,n}\|_{L^1[0,\infty)} \\ &\qquad\qquad + \left(\frac{\gamma\mu_3}{\widetilde{\gamma}^2} + \frac{\mu_1\mu_2}{\alpha\widetilde{\gamma}}\right)\sum_{n=1}^\infty \|z_{2,n}\|_{L^1[0,\infty)} + \frac{\mu_3}{\widetilde{\gamma}}\sum_{n=1}^\infty \|z_{3,n}\|_{L^1[0,\infty)} \\ &\qquad \leq \frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{1,0}| + \frac{\alpha\gamma\mu_3 + \widetilde{\gamma}^2\mu_1 + \widetilde{\gamma}\mu_1\mu_2}{\alpha\widetilde{\gamma}^2}\sum_{n=1}^\infty |b_n| + \frac{2\mu_1}{\widetilde{\gamma}}\sum_{n=1}^\infty \|z_{1,n}\|_{L^1[0,\infty)} \\ &\qquad\qquad\qquad + \frac{\alpha\gamma\mu_3 + \widetilde{\gamma}\mu_1\mu_2}{\alpha\widetilde{\gamma}^2}\sum_{n=1}^\infty \|z_{2,n}\|_{L^1[0,\infty)} + \frac{\mu_3}{\widetilde{\gamma}}\|z_3\|_Y. \end{aligned} \tag{51}$$

By (36) we obtain

$$\sum_{n=1}^\infty |c_n| \leq \frac{\gamma}{\widetilde{\gamma}}\sum_{n=1}^\infty |b_n| + \frac{\gamma}{\widetilde{\gamma}}\sum_{n=1}^\infty \|z_{2,n}\|_{L^1[0,\infty)}. \tag{52}$$

From (27)-(31) and the Fubini Theorem we have

$$\begin{aligned}
\|p_{2,n}\|_{L^1[0,\infty)} &\le \int_0^\infty |b_n|e^{-\widetilde{\gamma}x} + \int_0^\infty e^{-\widetilde{\gamma}x}\int_0^x |z_{2,n}|e^{\widetilde{\gamma}\tau}\mathrm{d}\tau dx \\
&= \frac{1}{\widetilde{\gamma}}|b_n| + \int_0^\infty |z_{2,n}|e^{\widetilde{\gamma}\tau}\int_\tau^\infty e^{-\widetilde{\gamma}x}\mathrm{d}\tau\mathrm{d}x \\
&= \frac{1}{\widetilde{\gamma}}|b_n| + \frac{1}{\widetilde{\gamma}}\|z_{2,n}\|_{L^1[0,\infty)}, \quad n\ge 1. \\
\|p_{1,n}\|_{L^1[0,\infty)} &\le \frac{1}{\widetilde{\gamma}}|a_n| + \frac{1}{\widetilde{\gamma}}\|z_{1,n}\|_{L^1[0,\infty)}, \quad n\ge 1. \\
\|p_{3,n}\|_{L^1[0,\infty)} &\le \frac{1}{\widetilde{\gamma}}|c_n| + \frac{1}{\widetilde{\gamma}}\|z_{3,n}\|_{L^1[0,\infty)}, \quad n\ge 1. \\
&\Longrightarrow \\
\|(p_1,p_2,p_3)\| &= \|p_1\|_X + \|p_2\|_X + \|p_3\|_Y \\
&= |p_{1,0}| + \sum_{n=1}^\infty \|p_{1,n}\|_{L^1[0,\infty)} + |p_{2,0}| \\
&\quad + \sum_{n=1}^\infty \|p_{2,n}\|_{L^1[0,\infty)} + \sum_{n=1}^\infty \|p_{3,n}\|_{L^1[0,\infty)} \\
&= \frac{1}{\widetilde{\gamma}+\lambda}|z_{1,0}| + \frac{1}{\widetilde{\gamma}+\lambda}|z_{2,0}| + \frac{1}{\widetilde{\gamma}}\sum_{n=1}^\infty (|a_n| + |b_n| + |c_n|) \\
&\quad + \frac{1}{\widetilde{\gamma}}\sum_{n=1}^\infty \left(\|z_{1,n}\|_{L^1[0,\infty)} + \|z_{2,n}\|_{L^1[0,\infty)} + \|z_{3,n}\|_{L^1[0,\infty)}\right) \\
&= \frac{1}{\widetilde{\gamma}+\lambda}|z_{1,0}| + \frac{1}{\widetilde{\gamma}+\lambda}|z_{2,0}| + \frac{1}{\widetilde{\gamma}}\sum_{n=1}^\infty (|a_n| + |b_n| + |c_n|) \\
&\quad + \frac{1}{\widetilde{\gamma}}\sum_{n=1}^\infty \|z_{1,n}\|_{L^1[0,\infty)} + \frac{1}{\widetilde{\gamma}}\sum_{n=1}^\infty \|z_{2,n}\|_{L^1[0,\infty)} + \frac{1}{\widetilde{\gamma}}\|z_3\|_Y. \qquad (53)
\end{aligned}$$

From which together with (48), (51) and (52) we estimate

$$\begin{aligned}
&\frac{1}{\widetilde{\gamma}}\left\{\sum_{n=1}^\infty |a_n| + \sum_{n=1}^\infty |b_n| + \sum_{n=1}^\infty |c_n|\right\} \\
&\le \frac{1}{\widetilde{\gamma}}\left\{\frac{\lambda}{\widetilde{\gamma}+\lambda}|z_{1,0}| + \frac{\alpha\gamma\mu_3 + \widetilde{\gamma}^2\mu_1 + \widetilde{\gamma}\mu_1\mu_2}{\alpha\widetilde{\gamma}^2}\sum_{n=1}^\infty |b_n|\right. \\
&\quad + \frac{2\mu_1}{\widetilde{\gamma}}\sum_{n=1}^\infty \|z_{1,n}\|_{L^1[0,\infty)} + \frac{\alpha\gamma\mu_3 + \widetilde{\gamma}\mu_1\mu_2}{\alpha\widetilde{\gamma}^2}\sum_{n=1}^\infty \|z_{2,n}\|_{L^1[0,\infty)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu_3}{\widetilde{\gamma}}\|z_3\|_Y+\sum_{n=1}^{\infty}|b_n|+\frac{\gamma}{\widetilde{\gamma}}\sum_{n=1}^{\infty}|b_n|+\frac{\gamma}{\widetilde{\gamma}}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}\Bigg\}\\
=&\frac{\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}|z_{1,0}|+\frac{\alpha\gamma\mu_3+\widetilde{\gamma}^2\mu_1+\widetilde{\gamma}\mu_1\mu_2+\alpha\widetilde{\gamma}^2+\alpha\gamma\widetilde{\gamma}}{\alpha\widetilde{\gamma}^3}\sum_{n=1}^{\infty}|b_n|\\
&+\frac{2\mu_1}{\widetilde{\gamma}^2}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)}+\frac{\alpha\gamma\mu_3+\widetilde{\gamma}\mu_1\mu_2+\alpha\gamma\widetilde{\gamma}}{\alpha\widetilde{\gamma}^3}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}+\frac{\mu_3}{\widetilde{\gamma}^2}\|z_3\|_Y\\
=&\frac{\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}|z_{1,0}|+\frac{\alpha\gamma\mu_3+(\mu_1+\alpha)\widetilde{\gamma}^2+(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}}{\alpha\widetilde{\gamma}^3}\\
&\times\Bigg\{\frac{\lambda\widetilde{\gamma}^3}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{2,0}|+\frac{\alpha\lambda\widetilde{\gamma}^2}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{1,0}|\\
&+\frac{8\widetilde{\gamma}^4(\alpha\widetilde{\gamma}+2\alpha\mu_1)}{(\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3)(7\widetilde{\gamma}^3+\alpha\gamma\mu_3)}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)}\\
&+\frac{8\widetilde{\gamma}^3(\mu_2\widetilde{\gamma}^2+\mu_1\mu_2\widetilde{\gamma}+\alpha\gamma\mu_3)}{(\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3)(7\widetilde{\gamma}^3+\alpha\gamma\mu_3)}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}\\
&+\frac{8\widetilde{\gamma}^4\alpha\mu_3}{(\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3)(7\widetilde{\gamma}^3+\alpha\gamma\mu_3)}\|z_3\|_Y\Bigg\}\\
&+\frac{2\mu_1}{\widetilde{\gamma}^2}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)}+\frac{\alpha\gamma\mu_3+(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}}{\alpha\widetilde{\gamma}^3}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}+\frac{\mu_3}{\widetilde{\gamma}^2}\|z_3\|_Y\\
=&\frac{\lambda}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)}|z_{1,0}|+\frac{\lambda\alpha\gamma\mu_3+\lambda(\mu_1+\alpha)\widetilde{\gamma}^2+\lambda(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}}{\alpha(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{2,0}|\\
&+\frac{\lambda\alpha\gamma\mu_3+\lambda(\mu_1+\alpha)\widetilde{\gamma}^2+\lambda(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{1,0}|\\
&+\frac{8\widetilde{\gamma}(\widetilde{\gamma}+2\mu_1)[\alpha\gamma\mu_3+(\mu_1+\alpha)\widetilde{\gamma}^2+(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}]}{[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)}\\
&+\frac{8(\mu_2\widetilde{\gamma}^2+\mu_1\mu_2\widetilde{\gamma}+\alpha\gamma\mu_3)[\alpha\gamma\mu_3+(\mu_1+\alpha)\widetilde{\gamma}^2+(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}]}{\alpha[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}\\
&+\frac{8\widetilde{\gamma}\mu_3[\alpha\gamma\mu_3+(\mu_1+\alpha)\widetilde{\gamma}^2+(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}]}{[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)}\|z_3\|_Y\\
&+\frac{2\mu_1}{\widetilde{\gamma}^2}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)}+\frac{\alpha\gamma\mu_3+(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}}{\alpha\widetilde{\gamma}^3}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}+\frac{\mu_3}{\widetilde{\gamma}^2}\|z_3\|_Y\\
=&\frac{\lambda(\widetilde{\gamma}^3-\alpha\gamma\mu_3)+\lambda\alpha\gamma\mu_3+\lambda(\mu_1+\alpha)\widetilde{\gamma}^2+\lambda(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}}{\widetilde{\gamma}(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{1,0}|\\
&+\frac{\lambda\alpha\gamma\mu_3+\lambda(\mu_1+\alpha)\widetilde{\gamma}^2+\lambda(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}}{\alpha(\widetilde{\gamma}+\alpha)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{2,0}|
\end{aligned}
$$

$$
\begin{aligned}
&+\{\{8\widetilde{\gamma}^3(\widetilde{\gamma}+2\mu_1)[\alpha\gamma\mu_3+(\mu_1+\alpha)\widetilde{\gamma}^2+(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}] \\
&\quad+2\mu_1[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\} \\
&\quad/\{\widetilde{\gamma}^2[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\}\}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)} \\
&+\{\{8\widetilde{\gamma}^3(\mu_2\widetilde{\gamma}^2+\mu_1\mu_2\widetilde{\gamma}+\alpha\gamma\mu_3)[\alpha\gamma\mu_3+(\mu_1+\alpha)\widetilde{\gamma}^2+(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}] \\
&+[\alpha\gamma\mu_3+(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}][\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\} \\
&/\{\alpha\widetilde{\gamma}^3[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\}\}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)} \\
&+\{\{8\widetilde{\gamma}^3\mu_3[\alpha\gamma\mu_3+(\mu_1+\alpha)\widetilde{\gamma}^2+(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}] \\
&\quad+\mu_3[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\} \\
&\quad\quad/\{\widetilde{\gamma}^2[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\}\}\|z_3\|_Y \\
&=\frac{\lambda\widetilde{\gamma}^2+\lambda(\mu_1+\alpha)\widetilde{\gamma}+\lambda(\mu_1\mu_2+\alpha\gamma)}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{1,0}| \\
&\quad+\frac{\lambda\alpha\gamma\mu_3+\lambda(\mu_1+\alpha)\widetilde{\gamma}^2+\lambda(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}}{\alpha(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{2,0}| \\
&\quad+\{\{(22\mu_1+8\alpha)\widetilde{\gamma}^6+(8\alpha\gamma+16\mu_1\alpha-12\mu_1^2-20\mu_1\mu_2)\widetilde{\gamma}^5 \\
&+[8\alpha\gamma\mu_3+16\mu_1(\mu_1\mu_2+\alpha\gamma)]\widetilde{\gamma}^4 \\
&\quad+4\alpha\gamma\mu_1\mu_3\widetilde{\gamma}^3-4(\mu_1+\mu_2)\alpha\gamma\mu_1\mu_3\widetilde{\gamma}^2-2(\alpha\gamma\mu_3)^2\mu_1\} \\
&\quad\quad/\{\widetilde{\gamma}^2[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\}\}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)} \\
&\quad+\{\{(15\mu_1\mu_2+8\mu_2\alpha+7\alpha\gamma)\widetilde{\gamma}^7+(8\alpha\mu_1\mu_2-7\mu_1\mu_2^2-7\mu_1^2\mu_2-7\alpha\gamma\mu_2-14\alpha\gamma\mu_1)\widetilde{\gamma}^6 \\
&\quad+[8(\mu_1\mu_2)^2+8\alpha\gamma\mu_1\mu_2+8\alpha^2\gamma\mu_3-6\alpha\gamma\mu_1\mu_3-6\alpha\gamma\mu_2\mu_3]\widetilde{\gamma}^5+[24\alpha\gamma\mu_1\mu_2\mu_3 \\
&+(\alpha\gamma)^2\mu_3]\widetilde{\gamma}^4 \\
&\quad+[2(\alpha\gamma\mu_3)^2-2\alpha\gamma\mu_1^2\mu_2\mu_3-2\alpha\gamma\mu_1\mu_3-2\alpha\gamma\mu_1\mu_2^2\mu_3-2(\alpha\gamma)^2\mu_2\mu_3]\widetilde{\gamma}^3 \\
&\quad-2(\alpha\gamma\mu_3)^2(\mu_1+\mu_2)\widetilde{\gamma}^2-(\alpha\gamma\mu_3)^2(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}-(\alpha\gamma\mu_3)^3\} \\
&\quad\quad/\{\alpha\widetilde{\gamma}^3[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\}\}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)} \\
&\quad+\{\{7\mu_3\widetilde{\gamma}^6+[8\mu_3\alpha-7\mu_1\mu_3-14\mu_2\mu_3]\widetilde{\gamma}^5+8\mu_3(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}^4+2\alpha\gamma\mu_3^2\widetilde{\gamma}^3 \\
&\quad-2\alpha\gamma\mu_3^2(\mu_1+\mu_2)\widetilde{\gamma}^2-(\alpha\gamma)^2\mu_3^3\} \\
&\quad\quad/\{\widetilde{\gamma}^2[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\}\}\|z_3\|_Y. \qquad (54)
\end{aligned}
$$

Through inserting (54) into (53) we obtain, by tedious calculation,

$$
\|(p_1,p_2,p_3)\|\le\frac{1}{\widetilde{\gamma}+\lambda}|z_{1,0}|+\frac{1}{\widetilde{\gamma}+\lambda}|z_{2,0}|+\frac{1}{\widetilde{\gamma}}\sum_{n=1}^{\infty}(|a_n|+|b_n|+|c_n|)
$$

$$
\begin{aligned}
&+\frac{1}{\widetilde{\gamma}}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)}+\frac{1}{\widetilde{\gamma}}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}+\frac{1}{\widetilde{\gamma}}\|z_3\|_Y\\
&\le\frac{1}{\widetilde{\gamma}+\lambda}|z_{1,0}|+\frac{1}{\widetilde{\gamma}+\lambda}|z_{2,0}|\\
&+\frac{\lambda\widetilde{\gamma}^2+\lambda(\mu_1+\alpha)\widetilde{\gamma}+\lambda(\mu_1\mu_2+\alpha\gamma)}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{1,0}|\\
&+\frac{\lambda\alpha\gamma\mu_3+\lambda(\mu_1+\alpha)\widetilde{\gamma}^2+\lambda(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}}{\alpha(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{2,0}|\\
&+\{\{(22\mu_1+8\alpha)\widetilde{\gamma}^6+(8\alpha\gamma+16\mu_1\alpha-12\mu_1^2-20\mu_1\mu_2)\widetilde{\gamma}^5\\
&+[8\alpha\gamma\mu_3+16\mu_1(\mu_1\mu_2+\alpha\gamma)]\widetilde{\gamma}^4\\
&+4\alpha\gamma\mu_1\mu_3\widetilde{\gamma}^3-4(\mu_1+\mu_2)\alpha\gamma\mu_1\mu_3\widetilde{\gamma}^2-2(\alpha\gamma\mu_3)^2\mu_1\}\\
&/\{\widetilde{\gamma}^2[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\}\}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)}\\
&+\{\{(15\mu_1\mu_2+8\mu_2\alpha+7\alpha\gamma)\widetilde{\gamma}^7\\
&+(8\alpha\mu_1\mu_2-7\mu_1\mu_2^2-7\mu_1^2\mu_2-7\alpha\gamma\mu_2-14\alpha\gamma\mu_1)\widetilde{\gamma}^6\\
&+[8(\mu_1\mu_2)^2+8\alpha\gamma\mu_1\mu_2+8\alpha^2\gamma\mu_3-6\alpha\gamma\mu_1\mu_3-6\alpha\gamma\mu_2\mu_3]\widetilde{\gamma}^5\\
&+[24\alpha\gamma\mu_1\mu_2\mu_3+(\alpha\gamma)^2\mu_3]\widetilde{\gamma}^4\\
&+[2(\alpha\gamma\mu_3)^2-2\alpha\gamma\mu_1^2\mu_2\mu_3-2\alpha\gamma\mu_1\mu_3-2\alpha\gamma\mu_1\mu_2^2\mu_3-2(\alpha\gamma)^2\mu_2\mu_3]\widetilde{\gamma}^3\\
&-2(\alpha\gamma\mu_3)^2(\mu_1+\mu_2)\widetilde{\gamma}^2-(\alpha\gamma\mu_3)^2(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}-(\alpha\gamma\mu_3)^3\}\\
&/\{\alpha\widetilde{\gamma}^3[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\}\}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}\\
&+\{\{7\mu_3\widetilde{\gamma}^6+[8\mu_3\alpha-7\mu_1\mu_3-14\mu_2\mu_3]\widetilde{\gamma}^5+8\mu_3(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}^4+2\alpha\gamma\mu_3^2\widetilde{\gamma}^3\\
&-2\alpha\gamma\mu_3^2(\mu_1+\mu_2)\widetilde{\gamma}^2-(\alpha\gamma)^2\mu_3^3\}\\
&/\{\widetilde{\gamma}^2[\widetilde{\gamma}^3-2(\mu_1+\mu_2)\widetilde{\gamma}^2-\alpha\gamma\mu_3](7\widetilde{\gamma}^3+\alpha\gamma\mu_3)\}\}\|z_3\|_Y\\
&+\frac{1}{\widetilde{\gamma}}\sum_{n=1}^{\infty}\|z_{1,n}\|_{L^1[0,\infty)}+\frac{1}{\widetilde{\gamma}}\sum_{n=1}^{\infty}\|z_{2,n}\|_{L^1[0,\infty)}+\frac{1}{\widetilde{\gamma}}\|z_3\|_Y\\
&=\frac{\widetilde{\gamma}^3+\lambda\widetilde{\gamma}^2+\lambda(\mu_1+\alpha)\widetilde{\gamma}+\lambda(\mu_1\mu_2+\alpha\gamma)-\alpha\gamma\mu_3}{(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{1,0}|\\
&+\frac{\alpha\widetilde{\gamma}^3+\lambda(\mu_1+\alpha)\widetilde{\gamma}^2+\lambda(\mu_1\mu_2+\alpha\gamma)\widetilde{\gamma}+\lambda\alpha\gamma\mu_3-\alpha^2\gamma\mu_3}{\alpha(\widetilde{\gamma}+\lambda)(\widetilde{\gamma}^3-\alpha\gamma\mu_3)}|z_{2,0}|\\
&+\{\{7\widetilde{\gamma}^7+(8\mu_1+8\alpha-14\mu_2)\widetilde{\gamma}^6
\end{aligned}
$$

$$
\begin{aligned}
&+ (8\alpha\gamma + 16\mu_1\alpha - 12\mu_1^2 - 20\mu_1\mu_2)\widetilde{\gamma}^5 + [2\alpha\gamma\mu_3 + 16\mu_1(\mu_1\mu_2 + \alpha\gamma)]\widetilde{\gamma}^4 \\
&+ (2\alpha\gamma\mu_1\mu_3 - 2\alpha\gamma\mu_2\mu_3)\widetilde{\gamma}^3 - 4(\mu_1 + \mu_2)\alpha\gamma\mu_1\mu_3\widetilde{\gamma}^2 - (\alpha\gamma\mu_3)^2\widetilde{\gamma} \\
&- 2(\alpha\gamma\mu_3)^2\mu_1\}/\{\widetilde{\gamma}^2[\widetilde{\gamma}^3 - 2(\mu_1 + \mu_2)\widetilde{\gamma}^2 - \alpha\gamma\mu_3](7\widetilde{\gamma}^3 + \alpha\gamma\mu_3)\}\} \sum_{n=1}^{\infty} \|z_{1,n}\|_{L^1[0,\infty)} \\
&+ \{\{7\alpha\widetilde{\gamma}^8 + (15\mu_1\mu_2 - 6\mu_2\alpha - 14\alpha\mu_1 + 7\alpha\gamma)\widetilde{\gamma}^7 \\
&+ (8\alpha\mu_1\mu_2 - 7\mu_1\mu_2^2 - 7\mu_1^2\mu_2 - 7\alpha\gamma\mu_2 - 14\alpha\gamma\mu_1)\widetilde{\gamma}^6 \\
&+ [8(\mu_1\mu_2)^2 + 8\alpha\gamma\mu_1\mu_2 + 2\alpha^2\gamma\mu_3 - 6\alpha\gamma\mu_1\mu_3 - 6\alpha\gamma\mu_2\mu_3]\widetilde{\gamma}^5 \\
&+ [24\alpha\gamma\mu_1\mu_2\mu_3 + (\alpha\gamma)^2\mu_3 - 2\alpha^2\gamma\mu_3(\mu_1 + \mu_2)]\widetilde{\gamma}^4 \\
&+ [2(\alpha\gamma\mu_3)^2 - 2\alpha\gamma\mu_1^2\mu_2\mu_3 - 2\alpha\gamma\mu_1\mu_3 - 2\alpha\gamma\mu_1\mu_2^2\mu_3 - 2(\alpha\gamma)^2\mu_2\mu_3]\widetilde{\gamma}^3 \\
&- (\alpha\gamma\mu_3)^2(\alpha + 2\mu_1 + 2\mu_2)\widetilde{\gamma}^2 - (\alpha\gamma\mu_3)^2(\mu_1\mu_2 + \alpha\gamma)\widetilde{\gamma} - (\alpha\gamma\mu_3)^3\} \\
&/\{\alpha\widetilde{\gamma}^3[\widetilde{\gamma}^3 - 2(\mu_1 + \mu_2)\widetilde{\gamma}^2 - \alpha\gamma\mu_3](7\widetilde{\gamma}^3 + \alpha\gamma\mu_3)\}\} \sum_{n=1}^{\infty} \|z_{2,n}\|_{L^1[0,\infty)} \\
&+ \{\{7\widetilde{\gamma}^7 + (7\mu_3 - 14\mu_1 - 14\mu_2)\widetilde{\gamma}^6 + (8\alpha\mu_3 - 7\mu_1\mu_3 - 14\mu_2\mu_3)\widetilde{\gamma}^5 \\
&+ (8\mu_1\mu_2\mu_3 + 2\alpha\gamma\mu_3)\widetilde{\gamma}^4 + (2\alpha\gamma\mu_3^2 - 2\alpha\gamma\mu_2\mu_3 - 2\alpha\gamma\mu_2\mu_3)\widetilde{\gamma}^3 \\
&- 2\alpha\gamma\mu_3^2(\mu_1 + \mu_2)\widetilde{\gamma}^2 - (\alpha\gamma\mu_3)^2\widetilde{\gamma} - (\alpha\gamma)^2\mu_3^3\} \\
&/\{\widetilde{\gamma}^2[\widetilde{\gamma}^3 - 2(\mu_1 + \mu_2)\widetilde{\gamma}^2 - \alpha\gamma\mu_3](7\widetilde{\gamma}^3 + \alpha\gamma\mu_3)\}\}\|z_3\|_Y \\
&\leq \frac{1}{\widetilde{\gamma} - (\lambda + \alpha + \mu_2 + \mu_3 + 2\alpha + 4\mu_1 + \frac{\lambda\mu_1}{\alpha} + \frac{3\mu_1\mu_2}{\alpha})} \\
&\times \{|z_{1,0}| + |z_{2,0}| + \sum_{n=1}^{\infty} \|z_{1,n}\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty} \|z_{2,n}\|_{L^1[0,\infty)} + \|z_3\|_Y\} \\
&= \frac{1}{\widetilde{\gamma} - (\lambda + \alpha + \mu_2 + \mu_3 + 2\alpha + 4\mu_1 + \frac{\lambda\mu_1}{\alpha} + \frac{3\mu_1\mu_2}{\alpha})}\|(z_1, z_2, z_3)\|.
\end{aligned}
$$

Which shows that

$$
\begin{aligned}
&(\widetilde{\gamma}I - A)^{-1} : X \longrightarrow X, \\
&\|(\widetilde{\gamma}I - A)^{-1}\| \leq \frac{1}{\widetilde{\gamma} - (\lambda + \alpha + \mu_2 + \mu_3 + 2\alpha + 4\mu_1 + \frac{\lambda\mu_1}{\alpha} + \frac{3\mu_1\mu_2}{\alpha})},
\end{aligned}
$$

when

$$
\widetilde{\gamma} > \max\left\{2(\mu_1 + \mu_2) + \alpha + 1, \alpha\gamma\mu_3, \sqrt{\gamma\mu_3}, \lambda + \gamma + \mu_2 + \mu_3 + 2\alpha + 4\mu_1 + \frac{\lambda\mu_1}{\alpha} + \frac{3\mu_1\mu_2}{\alpha}\right\}.
$$

In the second step we will prove that $D(A)$ is dense in X. From definition of $X \times Y$ we know that

$$\forall (p_1,p_2,p_3) \in X \times Y \implies |p_{1,0}| + \sum_{n=1}^{\infty} \|p_{1,n}\|_{L^1[0,\infty]} + |p_{2,0}| + \sum_{n=1}^{\infty} \|p_{2,n}\|_{L^1[0,\infty]}$$

$$+ \sum_{n=1}^{\infty} \|p_{3,n}\|_{L^1[0,\infty]} < \infty.$$

Therefore for $\forall \varepsilon > 0$, there exists N such that

$$\sum_{n=N}^{\infty} \|p_{1,n}\|_{L^1[0,\infty]} < \varepsilon, \quad \sum_{n=N}^{\infty} \|p_{2,n}\|_{L^1[0,\infty]} < \varepsilon, \quad \sum_{n=N}^{\infty} \|p_{2,n}\|_{L^1[0,\infty]} < \varepsilon.$$

If we define

$$L = \left\{ (p_1,p_2,p_3) \;\middle|\; \begin{array}{l} p_1(x) = (p_{1,0}, p_{1,1}(x), p_{1,2}(x), \cdots, p_{1,N}(x), 0, 0, \cdots), \\ p_2(x) = (p_{2,0}, p_{2,1}(x), p_{2,2}(x), \cdots, p_{2,N}(x), 0, 0, \cdots), \\ p_3(x) = (p_{3,1}(x), p_{3,2}(x), p_{3,3}(x), \cdots, p_{3,N}(x), 0, 0, \cdots), \\ p_{1,0}, p_{2,0} \in R, p_{1,n}, p_{2,n}, p_{3,n} \in L^1[0,\infty),\ n = 1,2,\cdots,N; \\ N \text{ is a finite positive integer.} \end{array} \right\}$$

then L is dense in $X \times Y$. Let

$$Z = \left\{ (p_1,p_2,p_3) \;\middle|\; \begin{array}{l} p_1(x) = (p_{1,0}, p_{1,1}(x), p_{1,2}(x), \cdots, p_{1,m}(x), 0, 0, \cdots), \\ p_2(x) = (p_{2,0}, p_{2,1}(x), p_{2,2}(x), \cdots, p_{2,m}(x), 0, 0, \cdots), \\ p_3(x) = (p_{3,1}(x), p_{3,2}(x), p_{3,3}(x), \cdots, p_{3,m}(x), 0, 0, \cdots), \\ p_{1,i}, p_{2,i}, p_{1,i} \in C_0^{\infty}[0,\infty) \text{ and there exist positive } c_i, d_i, e_i \\ \text{such that } p_{1,i}(x) = 0, x \in [0,c_i];\ p_{2,i}(x) = 0,\ x \in [0,d_i]; \\ p_{3,i}(x) = 0,\ x \in [0,e_i];\quad i = 1,2,\cdots,m. \end{array} \right\}.$$

From Adams [1] we know that Z is dense in L. So, in order to prove that $D(A)$ is dense in $X \times Y$, it suffices to prove that $D(A)$ is dense in Z. Take $(p_1,p_2,p_3) \in Z$, then there exist $c_i, d_i\ e_i\ (i = 1,2,3,\cdots,l)$ such that

$$\begin{aligned} p_1(x) &= (p_{1,0}, p_{1,1}(x), p_{1,2}(x), \cdots, p_{1,l}(x), 0, 0, \cdots), \\ p_2(x) &= (p_{2,0}, p_{2,1}(x), p_{2,2}(x), \cdots, p_{2,l}(x), 0, 0, \cdots), \\ p_3(x) &= (p_{3,1}, p_{3,2}(x), p_{3,3}(x), \cdots, p_{3,l}(x), 0, 0, \cdots), \\ p_{1,i}(x) &= 0, \quad x \in [0,c_i]; \quad p_{2,i}(x) = 0, \quad x \in [0,d_i], \\ p_{3,k}(x) &= 0, \quad x \in [0,e_i], \quad i = 1,2,\cdots,l, \end{aligned}$$

which result

$$p_{1,i}(x) = p_{2,i}(x) = p_{3,i}(x) = 0, \quad x \in [0,2s], \quad i = 1,2,\cdots,l,$$

where $0 < 2s < \min\{c_1, c_2, \cdots, c_l, d_1, d_2, \cdots, d_l, e_1, e_2, \cdots, e_l\}$. Define

$$f_1^s(0) = (p_{10}, f_{1,1}^s(0), f_{1,2}^s(0), f_{1,3}^s, \cdots, f_{1,l}^s(0), 0, 0, \cdots)$$

$$
\begin{aligned}
&=\Big(p_{1,0}, \lambda p_{1,0} + \int_{2s}^{\infty} \mu_1(x) p_{1,2}(x)dx + \int_{2s}^{\infty} \beta(x) p_{3,1}(x)dx,\\
&\int_{2s}^{\infty} \mu_1(x) p_{1,3}(x)dx + \int_{2s}^{\infty} \beta(x) p_{3,2}(x)dx,\\
&\int_{2s}^{\infty} \mu_1(x) p_{1,4}(x)dx + \int_{2s}^{\infty} \beta(x) p_{3,3}(x)dx \cdots,\\
&\int_{2s}^{\infty} \mu_1(x) p_{1,l}(x)dx + \int_{2s}^{\infty} \beta(x) p_{3,l-1}(x)dx, \int_{2s}^{\infty} \beta(x) p_{3,l}, 0, 0, \cdots \Big).
\end{aligned}
$$

$$
\begin{aligned}
f_2^s(0) =&(p_{2,0}, f_{2,1}^s(0), f_{2,2}^s(0), f_{1,3}^s(0), \cdots, f_{2,l}^s(0), 0, 0, \cdots)\\
=&\Big(p_{2,0}, \lambda p_{2,0} + \int_{2s}^{\infty} \mu_2(x) p_{2,2}(x)dx + \alpha \int_{2s}^{\infty} p_{1,1}(x)dx,\\
&\int_{2s}^{\infty} \mu_2(x) p_{2,3}(x)dx + \alpha \int_{2s}^{\infty} p_{1,2}(x)dx,\\
&\cdots, \int_{2s}^{\infty} \mu_2(x) p_{2,l}(x)dx + \alpha \int_{2s}^{\infty} p_{1,l-1}(x)dx, \alpha \int_{2s}^{\infty} p_{1,l}, 0, 0, \cdots, \Big).
\end{aligned}
$$

$$
\begin{aligned}
f_3^s(0) =&(f_{3,1}^s(0), f_{3,2}^s(0), f_{3,3}^s(0), \cdots, f_{3,l}^s(0), 0, 0, \cdots)\\
=&\Big(\gamma \int_{2s}^{\infty} p_{2,1}(x)dx, \gamma \int_{2s}^{\infty} p_{2,2}(x)dx, \cdots\\
&\gamma \int_{2s}^{\infty} p_{2,l-1}dx, \gamma \int_{2s}^{\infty} p_{2,l}dx, 0, 0, \cdots \Big)
\end{aligned}
$$

If we introduce

$$
\begin{aligned}
f_1^s(x) =&(p_{1,0}, f_{1,1}^s(x), f_{1,2}^s(x), f_{1,3}^s(x), \cdots, f_{1,l}^s(x), 0, 0, \cdots),\\
f_2^s(x) =&(p_{2,0}, f_{1,1}^s(x), f_{2,2}^s(x), f_{2,3}^s(x), \cdots, f_{2,l}^s(x), 0, 0, \cdots),\\
f_3^s(x) =&(f_{3,1}^s(x), f_{3,2}^s(x), f_{3,3}^s(x), \cdots, f_{3,l}^s(x) 0, 0, \cdots),
\end{aligned}
$$

where

$$
f_{1,i}^s(x) = \begin{cases} f_{1,i}^s(0)(1-\frac{x}{s})^2 & x \in [0,s],\\ -u_i(x-s)^2(x-2s)^2 & x \in [s,2s], \quad i = 1,2,3,\cdots,l;\\ p_{1,i}(x) & x \in [2s,\infty), \end{cases}
$$

$$
f_{2,i}^s(x) = \begin{cases} f_{2,i}^s(0)(1-\frac{x}{s})^2 & x \in [0,s],\\ -v_i(x-s)^2(x-2s)^2 & x \in [s,2s], \quad i = 1,2,3,\cdots,l;\\ p_{2,i}(x) & x \in [2s,\infty), \end{cases}
$$

$$f^s_{3,i}(x)=\begin{cases} f^s_{3,i}(0)(1-\frac{x}{s})^2 & x\in[0,s],\\ -w_i(x-s)^2(x-2s)^2 & x\in[s,2s], \quad i=1,2,3,\cdots,l;\\ p_{3,i}(x) & x\in[2s,\infty),\end{cases}$$

$$u_i=\frac{\int_0^s\mu_2(x)f^s_{2,i+1}(0)\left(1-\frac{x}{s}\right)^2dx+\alpha\int_0^s f^s_{1,i}(0)\left(1-\frac{x}{s}\right)^2dx}{\alpha\int_s^{2s}(x-s)^2(x-2s)^2dx}-\frac{\int_0^s f^s_{2,i+1}(0)\left(1-\frac{x}{s}\right)^2dx\int_s^{2s}\mu_2(x)(x-s)^2(x-2s)^2dx}{\alpha\left(\int_s^{2s}(x-s)^2(x-2s)^2dx\right)^2},\quad i=1,2,3,\cdots,l;$$

$$v_i=\frac{\int_0^s f^s_{2,i}(0)\left(1-\frac{x}{s}\right)^2dx}{\int_s^{2s}(x-s)^2(x-2s)^2dx},\quad i=1,2,\cdots,l;$$

$$\begin{aligned} w_i=&\frac{\int_0^s\mu_1(x)f^s_{1,i+1}(0)\left(1-\frac{x}{s}\right)^2dx+\int_0^s\beta(x)f^s_{3,i}(0)\left(1-\frac{x}{s}\right)^2dx}{\int_s^{2s}\beta(x)(x-s)^2(x-2s)^2dx}\\ &-\frac{\int_0^s\mu_2(x)f^s_{2,i+1}(0)\left(1-\frac{x}{s}\right)^2dx+\alpha\int_0^s f^s_{1,i}(0)\left(1-\frac{x}{s}\right)^2dx}{\alpha\int_s^{2s}(x-s)^2(x-2s)^2dx}\\ &\times\frac{\int_s^{2s}\mu_1(x)(x-s)^2(x-2s)^2dx}{\int_s^{2s}\beta(x)(x-s)^2(x-2s)^2dx}\\ &+\frac{\int_0^s f^s_{2,i+1}(0)\left(1-\frac{x}{s}\right)^2dx\int_s^{2s}\mu_2(x)(x-s)^2(x-2s)^2dx}{\alpha\left(\int_s^{2s}(x-s)^2(x-2s)^2dx\right)^2}\\ &\times\frac{\int_s^{2s}\mu_1(x)(x-s)^2(x-2s)^2dx}{\int_s^{2s}\beta(x)(x-s)^2(x-2s)^2dx},\quad i=1,2,\cdots,l.\end{aligned}$$

Then it is easy to verify that $(f^s_1,f^s_2,f^s_3)\in D(A)$. Moreover,

$$\begin{aligned}&\|(p_1,p_2,p_3)-(f^s_1,f^s_2,f^s_3)\|\\ =&\sum_{i=1}^{l}\int_0^\infty|p_{1,i}(x)-f^s_{1,i}(x)|dx+\sum_{i=1}^{l}\int_0^\infty|p_{2,i}(x)-f^s_{2,i}(x)|\\ &+\sum_{i=1}^{l}\int_0^\infty|p_{3,i}(x)-f^s_{3,i}(x)|\\ =&\sum_{i=1}^{l}\int_0^s|f^s_{1,i}(0)|\Big(1-\frac{s}{2}\Big)^2\mathrm{d}x+\sum_{i=1}^{l}\int_2^{2s}|u_i|(x-s)^2(x-2s)^2\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{l}\int_0^s |f_{2,i}^s(0)|\Big(1-\frac{s}{2}\Big)^2\mathrm{d}x+\sum_{i=1}^{l}\int_2^{2s}|v_i|(x-s)^2(x-2s)^2\mathrm{d}x\\
&+\sum_{i=1}^{l}\int_0^s |f_{3,i}^s(0)|\Big(1-\frac{s}{2}\Big)^2\mathrm{d}x+\sum_{i=1}^{l}\int_2^{2s}|w_i|(x-s)^2(x-2s)^2\mathrm{d}x\\
=&\sum_{i=1}^{l}|f_{1,i}^s(0)|\frac{s}{3}+\sum_{i=1}^{l}|u_i|\frac{s^5}{30}+\sum_{i=1}^{l}|f_{2,i}^s(0)|\frac{s}{3}+\sum_{i=1}^{l}|v_i|\frac{s^5}{30}\\
&+\sum_{i=1}^{l}|f_{3,i}^s(0)|\frac{s}{3}+\sum_{i=1}^{l}|w_i|\frac{s^5}{30}\to 0,\quad \text{as}\quad s\to 0.
\end{aligned}
$$

Which shows that $D(A)$ is dense in Z. In other words, $D(A)$ is dense in $X\times Y$. From the above two steps and the Hille-Yosida Theorem (see Gupur et al [5]) we know that A generates a C_0- semigroup.

In the following we will prove that U and E are bounded linear operators. From definitions of U and E we have, for $\forall(p_1,p_2,p_3)\in X\times Y$,

$$
\begin{aligned}
\|U(p_1,p_2,p_3)\|\le&\sum_{n=1}^{\infty}\int_0^{\infty}(\alpha+\lambda+\mu_1(x))|p_{1,n}(x)|\mathrm{d}x+\lambda\sum_{n=1}^{\infty}\int_0^{\infty}|p_{1,n}(x)|dx\\
&+\sum_{n=1}^{\infty}\int_0^{\infty}(\gamma+\lambda+\mu_2(x))|p_{2,n}(x)|\mathrm{d}x+\lambda\sum_{n=1}^{\infty}\int_0^{\infty}|p_{2,n}(x)|\mathrm{d}x\\
&+\sum_{n=1}^{\infty}\int_0^{\infty}(\lambda+\beta(x))|p_{3,n}(x)|\mathrm{d}x+\lambda\sum_{n=1}^{\infty}\int_0^{\infty}|p_{3,n}(x)|\mathrm{d}x\\
\le&(\alpha+\lambda+\mu_1)\sum_{n=1}^{\infty}\|p_{1,n}\|_{L^1[0,\infty)}+\lambda\sum_{n=1}^{\infty}\|p_{1,n}\|_{L^1[0,\infty)}\\
&+(\gamma+\lambda+\mu_2)\sum_{n=1}^{\infty}\|p_{2,n}\|_{L^1[0,\infty)}+\lambda\sum_{n=1}^{\infty}\|p_{2,n}\|_{L^1[0,\infty)}\\
&+(\lambda+\mu_3)\sum_{n=1}^{\infty}\|p_{3,n}\|_{L^1[0,\infty)}+\lambda\sum_{n=1}^{\infty}\|p_{3,n}\|_{L^1[0,\infty)}\\
\le&(\alpha+2\lambda+\mu_1)\|p_1\|_X+(\gamma+2\lambda+\mu_2)\|p_2\|_X+(2\lambda+\mu_3)\|p_3\|_Y\\
\le&\max\{\alpha+2\lambda+\mu_1,\gamma+2\lambda+\mu_2,2\lambda+\mu_3\}\|(p_1,p_2,p_3)\|,\\
\|E(p_1,p_2,p_3)\|\le&\int_0^{\infty}\mu_1(x)|p_{1,1}(x)|\mathrm{d}x+\int_0^{\infty}\mu_2(x)|p_{2,1}(x)|\mathrm{d}x\\
\le&\mu_1\|p_{1,n}\|_{L^1[0,\infty)}+\mu_2\|p_{2,n}\|_{L^1[0,\infty)}
\end{aligned}
$$

$$\leq \max\{\mu_1, \mu_2\}\|(p_1, p_2, p_3)\|.$$

Which shows that U and E are bounded operators. It is obvious that U and E are linear operators. Thus from the perturbation theory of C_0- semigroup we know that $A+U+E$ generates a C_0-semigroup $T(t)$ (see Gupur et al [5, p. 33]).

As the fourth step is concerned, we will prove that $T(t)$ is a positive contraction operator. For $(p_1, p_2, p_3) \in D(A)$ introduce

$$\phi_1(x) = (\frac{[p_{1,0}]^+}{p_{1,0}}, \frac{[p_{1,1}(x)]^+}{p_{1,1}(x)}, \frac{[p_{1,2}(x)]^+}{p_{1,2}(x)}, \cdots),$$
$$\phi_2(x) = (\frac{[p_{2,0}]^+}{p_{2,0}}, \frac{[p_{2,1}(x)]^+}{p_{2,1}(x)}, \frac{[p_{2,2}(x)]^+}{p_{2,2}(x)}, \cdots),$$
$$\phi_3(x) = (\frac{[p_{3,1}(x)]^+}{p_{3,1}(x)}, \frac{[p_{3,2}(x)]^+}{p_{3,2}(x)}, \frac{[p_{3,3}(x)]^+}{p_{3,3}(x)}, \cdots),$$

where

$$[p_{1,0}]^+ = \begin{cases} p_{1,0} & p_{1,0} > 0, \\ 0 & p_{1,0} \leq 0, \end{cases} \qquad [p_{1,n}(x)]^+ = \begin{cases} p_{1,n}(x) & p_{1,n}(x) > 0, \\ 0 & p_{1,n}(x) \leq 0, \end{cases}$$

$$[p_{2,0}]^+ = \begin{cases} p_{2,0} & p_{2,0} > 0, \\ 0 & p_{2,0} \leq 0, \end{cases} \qquad [p_{2,n}(x)]^+ = \begin{cases} p_{2,n}(x) & p_{2,n}(x) > 0, \\ 0 & p_{2,n}(x) \leq 0, \end{cases}$$

$$[p_{3,n}(x)]^+ = \begin{cases} p_{3,n}(x) & p_{3,n}(x) > 0, \\ 0 & p_{3,n}(x) \leq 0, \end{cases}$$

where $n = 1, 2, 3, \cdots$. If we define $V_n = \{x \in [0,\infty) \mid p_{1,n}(x) > 0\}$, $W_n = \{x \in [0,\infty) \mid p_{1,n}(x) \leq 0\}$, $n \geq 1$, then from $p_{i,n} \in L^1[0,\infty)$, $i = 1, 2, 3;$ $n \geq 1$, we have

$$\begin{aligned} &\int_0^\infty \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} \frac{[p_{1,n}(x)]^+}{p_{1,n}(x)} dx \\ &\quad = \int_{V_n} \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} \frac{[p_{1,n}(x)]^+}{p_{1,n}(x)} \mathrm{d}x + \int_{W_n} \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} \frac{[p_{1,n}(x)]^+}{p_{1,n}(x)} \mathrm{d}x \\ &\quad = \int_{V_n} \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} \mathrm{d}x = \int_0^\infty \frac{\mathrm{d}[p_{1,n}(x)]^+}{\mathrm{d}x} \mathrm{d}x = -[p_{1,n}(0)]^+, \quad n \geq 1. \end{aligned} \tag{55}$$

Similar way to (55) we deduce

$$\int_0^\infty \frac{\mathrm{d}p_{2,n}(x)}{\mathrm{d}x} \frac{[p_{2,n}(x)]^+}{p_{2,n}(x)} dx = -[p_{2,n}(0)]^+, \quad n \geq 1. \tag{56}$$

$$\int_0^\infty \frac{\mathrm{d}p_{3,n}(x)}{\mathrm{d}x} \frac{[p_{3,n}(x)]^+}{p_{3,n}(x)} dx = -[p_{3,n}(0)]^+, \quad n \geq 1. \tag{57}$$

By using boundary conditions on (p_1, p_2, p_3) we derive

$$\begin{aligned}\sum_{n=1}^{\infty}[p_{1,n}(0)]^+ &= [p_{1,1}(0)]^+ + \sum_{n=2}^{\infty}[p_{1,n}(0)]^+ \\ &\le \lambda[p_{1,0}]^+ + \sum_{n=1}^{\infty}\int_0^{\infty}\mu_1(x)[p_{1,n+1}(x)]^+\mathrm{d}x + \sum_{n=1}^{\infty}\int_0^{\infty}\beta(x)[p_{3,n}(x)]^+\mathrm{d}x, \quad (58)\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}[p_{2,n}(0)]^+ &= [p_{2,1}(0)]^+ + \sum_{n=2}^{\infty}[p_{2,n}(0)]^+ \\ &\le \lambda[p_{2,0}]^+ + \sum_{n=1}^{\infty}\int_0^{\infty}\mu_2(x)[p_{2,n+1}(x)]^+\mathrm{d}x + \alpha\sum_{n=1}^{\infty}\int_0^{\infty}[p_{1,n}(x)]^+\mathrm{d}x, \quad (59)\end{aligned}$$

$$\sum_{n=1}^{\infty}[p_{3,n}(0)]^+ \le \gamma\sum_{n=1}^{\infty}\int_0^{\infty}[p_{2,n}(x)]^+\mathrm{d}x. \quad (60)$$

By using (55)-(60) we estimate, for $(p_1, p_2, p_3) \in D(A)$, (ϕ_1, ϕ_2, ϕ_3),

$$\begin{aligned}&\langle (A+U+E)(p_1,p_2,p_3), (\phi_1,\phi_2,\phi_3)\rangle \\ =&\frac{[p_{1,0}]^+}{p_{1,0}}\left\{-\lambda p_{1,0} + \int_0^{\infty}\mu_1(x)p_{1,1}(x)\mathrm{d}x\right\} \\ &+ \int_0^{\infty}\left\{-\frac{\mathrm{d}p_{1,1}(x)}{\mathrm{d}x} - (\alpha+\lambda+\mu_1(x))p_{1,1}(x)\right\}\frac{[p_{1,1}(x)]^+}{p_{1,1}(x)}\mathrm{d}x \\ &+ \sum_{n=2}^{\infty}\int_0^{\infty}\left\{-\frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} - (\alpha+\lambda+\mu_1(x))p_{1,n}(x) + \lambda p_{1,n-1}(x)\right\}\frac{[p_{1,n}(x)]^+}{p_{1,n}(x)}\mathrm{d}x \\ &+ \frac{[p_{2,0}]^+}{p_{2,0}}\Big\{-\lambda p_{2,0} + \int_0^{\infty}\mu_2(x)p_{2,1}(x)\mathrm{d}x\Big\} \\ &+ \int_0^{\infty}\left\{-\frac{\mathrm{d}p_{2,1}(x)}{\mathrm{d}x} + (\gamma+\lambda+\mu_2(x))p_{2,1}(x)\right\}\frac{[p_{2,1}(x)]^+}{p_{2,1}(x)}\mathrm{d}x \\ &+ \sum_{n=2}^{\infty}\int_0^{\infty}\left\{-\frac{\mathrm{d}p_{2,n}(x)}{\mathrm{d}x} - (\gamma+\lambda+\mu_2(x))p_{2,n}(x) + \lambda p_{2,n-1}(x)\right\}\frac{[p_{2,n}(x)]^+}{p_{2,n}(x)}\mathrm{d}x \\ &+ \int_0^{\infty}\left\{-\frac{\mathrm{d}p_{3,1}(x)}{\mathrm{d}x} - (\lambda+\beta(x))p_{3,1}(x)\right\}\frac{[p_{3,1}(x)]^+}{p_{3,1}(x)}\mathrm{d}x \\ &+ \sum_{n=2}^{\infty}\int_0^{\infty}\left\{-\frac{\mathrm{d}p_{3,n}(x)}{\mathrm{d}x} - (\gamma+\lambda+\beta(x))p_{3,n}(x) + \lambda p_{3,n-1}(x)\right\}\frac{[p_{3,n}(x)]^+}{p_{3,n}(x)}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
=&-\lambda[p_{1,0}]^{+}+\frac{[p_{1,0}]^{+}}{p_{1,0}}\int_0^\infty \mu_1(x)p_{1,1}(x)\mathrm{d}x-\sum_{n=1}^{\infty}\int_0^\infty \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x}\frac{[p_{1,n}(x)]^{+}}{p_{1,n}(x)}\mathrm{d}x\\
&-\sum_{n=1}^{\infty}\int_0^\infty(\alpha+\lambda+\mu_1(x))[p_{1,n}(x)]^{+}\mathrm{d}x+\lambda\sum_{n=2}^{\infty}\int_0^\infty p_{1,n-1}(x)\frac{[p_{1,n}(x)]^{+}}{p_{1,n}(x)}\mathrm{d}x\\
&-\lambda[p_{2,0}]^{+}+\frac{[p_{2,0}]^{+}}{p_{2,0}}\int_0^\infty \mu_2(x)p_{2,1}(x)\mathrm{d}x-\sum_{n=1}^{\infty}\int_0^\infty \frac{\mathrm{d}p_{2,n}(x)}{\mathrm{d}x}\frac{[p_{2,n}(x)]^{+}}{p_{2,n}(x)}\mathrm{d}x\\
&-\sum_{n=1}^{\infty}\int_0^\infty(\gamma+\lambda+\mu_2(x))[p_{2,n}(x)]^{+}\mathrm{d}x+\lambda\sum_{n=2}^{\infty}\int_0^\infty p_{2,n-1}(x)\frac{[p_{2,n}(x)]^{+}}{p_{2,n}(x)}\mathrm{d}x\\
&-\sum_{n=1}^{\infty}\int_0^\infty \frac{\mathrm{d}p_{3,n}(x)}{\mathrm{d}x}\frac{[p_{3,n}(x)]^{+}}{p_{3,n}(x)}\mathrm{d}x-\sum_{n=1}^{\infty}\int_0^\infty(\lambda+\beta(x))[p_{3,n}(x)]^{+}\mathrm{d}x\\
&+\lambda\sum_{n=2}^{\infty}\int_0^\infty p_{3,n-1}(x)\frac{[p_{3,n}(x)]^{+}}{p_{3,n}(x)}\mathrm{d}x\\
=&-\lambda[p_{1,0}]^{+}+\frac{[p_{1,0}]^{+}}{p_{1,0}}\int_0^\infty \mu_1(x)p_{1,1}(x)\mathrm{d}x+\sum_{n=1}^{\infty}[p_{1,n}(0)]^{+}\\
&-\sum_{n=1}^{\infty}\int_0^\infty(\alpha+\lambda+\mu_1(x))[p_{1,n}(x)]^{+}\mathrm{d}x+\lambda\sum_{n=1}^{\infty}\int_0^\infty p_{1,n}(x)\frac{[p_{1,n+1}(x)]^{+}}{p_{1,n+1}(x)}\mathrm{d}x\\
&-\lambda[p_{2,0}]^{+}+\frac{[p_{2,0}]^{+}}{p_{2,0}}\int_0^\infty \mu_2(x)p_{2,1}(x)\mathrm{d}x+\sum_{n=1}^{\infty}[p_{2,n}(0)]^{+}\\
&-\sum_{n=1}^{\infty}\int_0^\infty(\gamma+\lambda+\mu_2(x))[p_{2,n}(x)]^{+}\mathrm{d}x+\lambda\sum_{n=1}^{\infty}\int_0^\infty p_{2,n}(x)\frac{[p_{2,n+1}(x)]^{+}}{p_{2,n+1}(x)}\mathrm{d}x\\
&+\sum_{n=1}^{\infty}[p_{3,n}(0)]^{+}-\sum_{n=1}^{\infty}\int_0^\infty(\lambda+\beta(x))[p_{3,n}(x)]^{+}\mathrm{d}x\\
&+\lambda\sum_{n=1}^{\infty}\int_0^\infty p_{3,n}(x)\frac{[p_{3,n+1}(x)]^{+}}{p_{3,n+1}(x)}\mathrm{d}x\\
\le&-\lambda[p_{1,0}]^{+}+\frac{[p_{1,0}]^{+}}{p_{1,0}}\int_0^\infty \mu_1(x)[p_{1,1}(x)]^{+}\mathrm{d}x+\lambda[p_{1,0}]^{+}\\
&+\sum_{n=1}^{\infty}\int_0^\infty \mu_1(x)[p_{1,n+1}(x)]^{+}\mathrm{d}x+\sum_{n=1}^{\infty}\int_0^\infty \beta(x)[p_{3,n}(x)]^{+}\mathrm{d}x\\
&-\sum_{n=1}^{\infty}\int_0^\infty(\alpha+\lambda+\mu_1(x))[p_{1,n}(x)]^{+}\mathrm{d}x+\lambda\sum_{n=1}^{\infty}\int_0^\infty [p_{1,n}(x)]^{+}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
& -\lambda[p_{2,0}]^+ + \frac{[p_{2,0}]^+}{p_{2,0}}\int_0^\infty \mu_2(x)[p_{2,1}(x)]^+\mathrm{d}x + \lambda[p_{2,0}]^+ \\
& + \sum_{n=1}^\infty \int_0^\infty \mu_2(x)[p_{2,n+1}(x)]^+\mathrm{d}x + \alpha\sum_{n=1}^\infty \int_0^\infty [p_{1,n}(x)]^+\mathrm{d}x \\
& - \sum_{n=1}^\infty \int_0^\infty (\gamma+\lambda+\mu_2(x))[p_{2,n}(x)]^+\mathrm{d}x + \lambda\sum_{n=1}^\infty \int_0^\infty [p_{2,n}(x)]^+\mathrm{d}x \\
& + \gamma\sum_{n=1}^\infty [p_{2,n}(x)]^+\mathrm{d}x - \sum_{n=1}^\infty \int_0^\infty (\lambda+\beta(x))[p_{3,n}(x)]^+\mathrm{d}x + \lambda\sum_{n=1}^\infty \int_0^\infty [p_{3,n}(x)]^+\mathrm{d}x \\
= & \left\{\frac{[p_{1,0}]^+}{p_{1,0}} - 1\right\}\int_0^\infty \mu_1(x)[p_{1,1}(x)]^+\mathrm{d}x \\
& + \left\{\frac{[p_{2,0}]^+}{p_{2,0}} - 1\right\}\int_0^\infty \mu_2(x)[p_{2,1}(x)]^+\mathrm{d}x \le 0. \qquad (61)
\end{aligned}
$$

In (61) we used the following inequalities.

$$
\int_0^\infty \mu_i(x)p_{1,1}(x)dx \le \int_0^\infty \mu_1(x)[p_{1,1}(x)]^+dx, \quad i=1,2.
$$

$$
\int_0^\infty \lambda p_{1,n}(x)\frac{[p_{1,n+1}(x)]^+}{p_{1,n+1}(x)}dx \le \int_0^\infty \lambda[p_{1,n}(x)]^+dx, \quad n \ge 1.
$$

(61) shows that $A+U+E$ is a dispersive operator. Therefore, from which together with the first step, second step and the Phillips Theorem we know that $A+U+E$ generates a positive contraction C_0-semigroup (see Gupur et al [5, p. 29]). From the uniqueness of C_0-semigroup it follows that this semigroup is just $T(t)$ (see Gupur et al [5, p. 10]). The proof of this theorem is complete. □

It is the same as Gupur [4], $X^*\times Y^*$, dual space of $X\times Y$, is as follows.

$$
X^*\times Y^* = \Big\{(p_1^*, p_2^*, p_3^*)\Big|
$$

$$
p_1^*, p_2^* \in X^*, p_3^* \in Y^*, \ \ |||(p_1^*, p_2^*, p_3^*)||| = \sup\left\{|||p_1^*|||_{X^*}, |||p_2^*|||_{X^*}, |||p_3^*|||_{Y^*}\right\}\Big\},
$$

$$
X^* = \Big\{p_1^*, p_2^* \in R\times L^\infty[0,\infty)\times L^\infty[0,\infty)\times\cdots\Big|
$$

$$
\|p_1^*\||_{X^*} = \sup_{n\ge 1}\left\{|p_{1,0}^*|, \|p_{1,n}^*\|_{L^\infty[0,\infty)}\right\} < \infty
$$

$$|||p_2^*|||_{X^*} = \sup_{n\geq 1}\left\{|p_{2,0}^*|, \|p_{2,n}^*\|_{L^\infty[0,\infty)}\right\} < \infty\Bigg\},$$

$Y^* =$

$$\left\{p_3^* \in L^\infty[0,\infty)\times L^\infty[0,\infty)\times\cdots \,\middle|\, |||p_3^*|||_{Y^*} = \sup_{n\geq 1}\left\{\|p_{3,n}^*\|_{L^\infty[0,\infty)}\right\} < \infty\right\}.$$

It is easy to verify that $X^*\times Y^*$ is a Banach space. In $X\times Y$ we introduce

$$\begin{aligned} K = \Bigg\{(p_1,p_2,p_3)\in X \Bigg| & p_1 = (p_{1,0}, p_{1,1}(x), p_{1,2}(x),\cdots),\ p_{1,0}\geq 0,\ p_{1,n}(x)\geq 0 \\ & p_2 = (p_{2,0}, p_{2,1}(x), p_{2,2}(x),\cdots),\ p_{2,0}\geq 0,\ p_{2,n}(x)\geq 0 \\ & p_3 = (p_{3,1}(x), p_{3,2}(x),\cdots),\ p_{3,n}(x)\geq 0,\ n\geq 1,\ x\in[0,\infty)\Bigg\}, \end{aligned}$$

then K is a cone in $X\times Y$. For $(p_1,p_2,p_3)\in D(A)\cap K$ we take

$$(p_1^*,p_2^*,p_3^*) = \|(p_1,p_2,p_3)\|\left(\begin{pmatrix}1\\1\\1\\\vdots\end{pmatrix}, \begin{pmatrix}1\\1\\1\\\vdots\end{pmatrix}, \begin{pmatrix}1\\1\\1\\\vdots\end{pmatrix}\right),$$

then

$$\begin{aligned} &\langle(p_1,p_2,p_3),(p_1^*,p_2^*,p_3^*)\rangle = \langle p_1,p_1^*\rangle + \langle p_2,p_2^*\rangle + \langle p_3,p_3^*\rangle \\ =&\|(p_1,p_2,p_3)\|p_{1,0} + \sum_{n=1}^{\infty}\|(p_1,p_2,p_3)\|\int_0^\infty p_{1,n}(x)dx \\ &+ \|(p_1,p_2,p_3)\|p_{2,0} + \sum_{n=1}^{\infty}\|(p_1,p_2,p_3)\|\int_0^\infty p_{2,n}(x)dx \\ &+ \sum_{n=1}^{\infty}\|(p_1,p_2,p_3)\|\int_0^\infty p_{3,n}(x)dx \\ =&\|(p_1,p_2,p_3)\|\left(p_{1,0} + \sum_{n=1}^{\infty}\int_0^\infty p_{1,n}(x)dx\right) \\ =&\|(p_1,p_2,p_3)\|\left(p_{2,0} + \sum_{n=1}^{\infty}\int_0^\infty p_{2,n}(x)dx\right) + \|(p_1,p_2,p_3)\|\|p_3\|_Y \\ =&\|(p_1,p_2,p_3)\|(\|p_1\|_X + \|p_2\|_X + \|p_3\|_Y) \\ =&\|(p_1,p_2,p_3)\|^2 = |||(p_1^*,p_2^*,p_3^*)|||^2, \end{aligned}$$

that is, $(p_1^*,p_2^*,p_3^*)\in\theta((p_1,p_2,p_3))$, where

$$\theta((p_1,p_2,p_3)) = \big\{(p_1^*,p_2^*,p_3^*) \in X^* \mid \langle (p_1,p_2,p_3),(p_1^*,p_2^*,p_3^*)\rangle = \|(p_1,p_2,p_3))\|^2 = \||(p_1^*,p_2^*,p_3^*)\||^2\big\}.$$

For such $(p_1,p_2,p_3) \in D(A)$ and $(p_1^*,p_2^*,p_3^*) \in \theta((p_1,p_2,p_3))$ we have

$$\begin{aligned}
&\langle (A+U+E)(p_1,p_2,p_3),(p_1^*,p_2^*,p_3^*)\rangle \\
&= \|(p_1,p_2,p_3)\|\left\{-\lambda p_{1,0} + \int_0^\infty \mu_1(x)p_{1,1}(x)dx\right\} \\
&+ \int_0^\infty \|(p_1,p_2,p_3)\|\left\{-\frac{\mathrm{d}p_{1,1}(x)}{\mathrm{d}x} - (\alpha+\lambda+\mu_1(x))p_{1,1}(x)\right\}\mathrm{d}x \\
&+\sum_{n=2}^\infty \int_0^\infty \|(p_1,p_2,p_3)\|\left\{-\frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x} - (\alpha+\lambda+\mu_1(x))p_{1,n}(x) + \lambda p_{1,n-1}(x)\right\}\mathrm{d}x \\
&+ \|(p_1,p_2,p_3)\|\left\{-\lambda p_{2,0} + \int_0^\infty \mu_2(x)p_{2,1}(x)dx\right\} \\
&+ \int_0^\infty \|(p_1,p_2,p_3)\|\left\{-\frac{\mathrm{d}p_{2,1}(x)}{\mathrm{d}x} - (\gamma+\lambda+\mu_2(x))p_{2,1}(x)\right\}\mathrm{d}x \\
&+\sum_{n=2}^\infty \int_0^\infty \|(p_1,p_2,p_3)\|\left\{-\frac{\mathrm{d}p_{2,n}(x)}{\mathrm{d}x} - (\gamma+\lambda+\mu_2(x))p_{2,n}(x) + \lambda p_{2,n-1}(x)\right\}\mathrm{d}x \\
&+ \int_0^\infty \|(p_1,p_2,p_3)\|\left\{-\frac{\mathrm{d}p_{3,1}(x)}{\mathrm{d}x} - (\lambda+\beta(x))p_{3,1}(x)\right\}\mathrm{d}x \\
&+\sum_{n=2}^\infty \int_0^\infty \|(p_1,p_2,p_3)\|\left\{-\frac{\mathrm{d}p_{3,n}(x)}{\mathrm{d}x} - (\lambda+\beta(x))p_{3n}(x) + \lambda p_{3,n-1}(x)\right\}\mathrm{d}x \\
&= \|(p_1,p_2,p_3)\|\left\{-\lambda p_{1,0} + \int_0^\infty \mu_1(x)p_{1,1}(x)dx\right\} \\
&-\|(p_1,p_2,p_3)\|\sum_{n=1}^\infty \int_0^\infty \frac{\mathrm{d}p_{1,n}(x)}{\mathrm{d}x}\mathrm{d}x - \|(p_1,p_2,p_3)\|\sum_{n=1}^\infty \int_0^\infty (\alpha+\lambda+\mu_1(x))p_{1,n}(x)\mathrm{d}x \\
&+ \|(p_1,p_2,p_3)\|\sum_{n=2}^\infty \int_0^\infty \lambda p_{1,n-1}(x)\mathrm{d}x \\
&+ \|(p_1,p_2,p_3)\|\left\{-\lambda p_{2,0} + \int_0^\infty \mu_2(x)p_{2,1}(x)dx\right\} \\
&-\|(p_1,p_2,p_3)\|\sum_{n=1}^\infty \int_0^\infty \frac{\mathrm{d}p_{2,n}(x)}{\mathrm{d}x}\mathrm{d}x - \|(p_1,p_2,p_3)\|\sum_{n=1}^\infty \int_0^\infty (\gamma+\lambda+\mu_2(x))p_{2,n}(x)\mathrm{d}x
\end{aligned}$$

$$+\|(p_1,p_2,p_3)\|\sum_{n=2}^{\infty}\int_0^{\infty}\lambda p_{2,n-1}(x)\mathrm{d}x$$

$$-\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\mathrm{d}p_{3,n}(x)}{\mathrm{d}x}\mathrm{d}x-\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}(\lambda+\beta(x))p_{3,n}(x)\mathrm{d}x$$

$$+\|(p_1,p_2,p_3)\|\sum_{n=2}^{\infty}\int_0^{\infty}\lambda p_{3,n-1}(x)\mathrm{d}x$$

$$=\|(p_1,p_2,p_3)\|\left\{-\lambda p_{1,0}+\int_0^{\infty}\mu_1(x)p_{1,1}(x)dx\right\}+\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}p_{1,n}(0)$$

$$-\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}(\alpha+\lambda+\mu_1(x))p_{1,n}(x)\mathrm{d}x$$

$$+\lambda\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}p_{1,n}(x)\mathrm{d}x$$

$$+\|(p_1,p_2,p_3)\|\left\{-\lambda p_{2,0}+\int_0^{\infty}\mu_2(x)p_{2,1}(x)dx\right\}+\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}p_{2,n}(0)$$

$$-\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}(\gamma+\lambda+\mu_2(x))p_{2,n}(x)\mathrm{d}x$$

$$+\lambda\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}p_{2,n}(x)\mathrm{d}x+\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}p_{3,n}(0)$$

$$-\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}(\lambda+\beta(x))p_{3,n}(x)\mathrm{d}x+\lambda\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}p_{3,n}(x)\mathrm{d}x$$

$$=\|(p_1,p_2,p_3)\|\left\{-\lambda p_{1,0}+\int_0^{\infty}\mu_1(x)p_{1,1}(x)dx\right\}$$

$$+\|(p_1,p_2,p_3)\|\left\{\lambda p_{1,0}+\sum_{n=1}^{\infty}\int_0^{\infty}\mu_1(x)p_{1,n+1}(x)\mathrm{d}x+\sum_{n=1}^{\infty}\beta(x)p_{3,n}(x)\mathrm{d}x\right\}$$

$$-\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}(\alpha+\lambda+\mu_1(x))p_{1,n}(x)\mathrm{d}x$$

$$+\lambda\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}p_{1,n}(x)\mathrm{d}x$$

$$+\|(p_1,p_2,p_3)\|\left\{-\lambda p_{2,0}+\int_0^{\infty}\mu_2(x)p_{2,1}(x)dx\right\}$$

$$
\begin{aligned}
&+\|(p_1,p_2,p_3)\|\left\{\lambda p_{2,0}+\sum_{n=1}^{\infty}\int_0^{\infty}\mu_2(x)p_{2,n+1}(x)\mathrm{d}x+\alpha\sum_{n=1}^{\infty}p_{1,n}(x)\mathrm{d}x\right\}\\
&-\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}(\gamma+\lambda+\mu_2(x))p_{2,n}(x)\mathrm{d}x\\
&+\lambda\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}p_{2,n}(x)\mathrm{d}x\\
&+\gamma\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}p_{2,n}(x)\mathrm{d}x-\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}(\lambda+\beta(x))p_{3,n}(x)\mathrm{d}x\\
&+\lambda\|(p_1,p_2,p_3)\|\sum_{n=1}^{\infty}\int_0^{\infty}p_{3,n}(x)\mathrm{d}x\\
&=\|(p_1,p_2,p_3)\|\left\{\int_0^{\infty}\mu_1(x)p_{1,1}(x)+\sum_{n=1}^{\infty}\int_0^{\infty}\mu_1(x)p_{1,n+1}(x)\mathrm{d}x\right.\\
&+\alpha\sum_{n=1}^{\infty}\int_0^{\infty}p_{1,n}(x)\mathrm{d}x+\lambda\sum_{n=1}^{\infty}\int_0^{\infty}p_{1,n}(x)\mathrm{d}x\\
&\left.-\sum_{n=1}^{\infty}\int_0^{\infty}(\alpha+\lambda+\mu_1(x))p_{1,n}\mathrm{d}x\right\}\\
&+\|(p_1,p_2,p_3)\|\left\{\int_0^{\infty}\mu_2(x)p_{2,1}(x)dx+\sum_{n=1}^{\infty}\int_0^{\infty}\mu_2(x)p_{2,n+1}(x)\mathrm{d}x\right.\\
&+\gamma\sum_{n=1}^{\infty}\int_0^{\infty}p_{2,n}(x)\mathrm{d}x+\lambda\sum_{n=1}^{\infty}\int_0^{\infty}p_{2,n}(x)\mathrm{d}x\\
&\left.-\sum_{n=1}^{\infty}\int_0^{\infty}(\gamma+\lambda+\mu_2(x))p_{2,n}\mathrm{d}x\right\}\\
&+\|(p_1,p_2,p_3)\|\left\{\sum_{n=1}^{\infty}\int_0^{\infty}\beta(x)p_{3,n}(x)\mathrm{d}x+\lambda\sum_{n=1}^{\infty}\int_0^{\infty}p_{3,n}(x)\mathrm{d}x\right.\\
&\left.-\sum_{n=1}^{\infty}\int_0^{\infty}(\lambda+\beta(x))p_{3,n}(x)\mathrm{d}x\right\}=0. \qquad (62)
\end{aligned}
$$

(62) shows that $A+U+E$ is conservative for $\theta(\cdot)$. Since the initial value $(p_1,p_2,p_3)(0)\in D(A^2)\cap K$, by using the Fattorini Theorem we derive the following result(see Fattorini [3], Gupur et al [5, p. 132]).

Theorem 2. *$T(t)$ is isometric for the initial value of the system (15)-(16),* *that is,*

$$\|T(t)(p_1, p_2, p_3)(0)\| = \|(p_1, p_2, p_3)(0)\|, \quad \forall t \in [0, \infty). \tag{63}$$

From Theorem 1 and Theorem 2 we obtain the desired result in this paper.

Theorem 3. *The system (15)-(16) has a unique positive time-dependent solution* $(p_1, p_2, p_3)(x, t)$, *which satisfies*

$$\|(p_1, p_2, p_3)(\cdot, t)\| = 1, \quad \forall t \in [0, \infty).$$

Proof. By Theorem 1 and Gupur et al [5, p. 35] we know that the system (15)-(16) has a unique positive time-dependent solution $(p_1, p_2, p_3)(x, t)$, which can be expressed as

$$(p_1, p_2, p_3)(x, t) = T(t)(p_1, p_2, p_3)(0), \quad t \in [0, \infty). \tag{64}$$

Since $(p_1, p_2, p_3)(0) \in D(A^2) \cap Y$, by Theorem 2 and (64) we have

$$\|(p_1, p_2, p_3)(\cdot, t)\| = \|T(t)(p_1, p_2, p_3)(0)\| = \|(p_1, p_2, p_3)(0)\| = 1, \quad \forall t \in [0, \infty). \tag{65}$$

The proof of this theorem is complete. □

(65) just reflects the physical background of the problem.

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