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ON THE GONALITY OF REDUCIBLE CURVES

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Abstract: Here we consider several definitions of gonality of a reduced curve X and study them when the dual graph of X (obtained only from the singular points of X lying on at least two irreducible components) is contractible. We assume that all singular points of X lying on more than one component are ordinary nodes.

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1. Introduction

Let X be a reduced and connected projective curve defined over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. We assume that every singular point of X lying on at least two irreducible components of X is an ordinary node of X. Set $g := p_a(X)$. Let $\mathcal{B}(X)$ denote the set of the irreducible components of X. The *dual graph* $\|X\|$ of X is the following non-oriented graph with multiple edges, but no loop. There is a bijection between the set of all vertices of $\|X\|$ and $\mathcal{B}(X)$. If $v \neq w$ are vertices of $\|X\|$ and T_v, T_w are the associated irreducible components of X, then v and w are connected by $\sharp(T_v \cap T_w)$ edges. Notice that the graph $\|X\|$ ignores the singularities of X lying on a unique irreducible component of X. For every union A of some of the irreducible components of X let $A^{[c]}$ denote the closure of $X \backslash A$ in X. Thus $(A^{[c]})^{[c]} = A$. For the elementary properties of depth 1 coherent sheaves on reduced curves, see [6], Parts VII and VIII. We say that a depth 1 sheaf F on X has pure pure rank 1 if its restriction to X_{reg} is a pure rank 1 vector bundle. Let F be sheaf on X with pure rank 1

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and with depth 1. Set $\mathrm{Sing}(F) := \{P \in X : F$ is not locally free at $P\}$. Hence $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$. The degree $\deg(F)$ of F may be defined by the Rieman-Roch formula. We say that F is *strongly positive* if $\deg((F|T)/\mathrm{Tors}(F|T)) > 0$. A line bundle is strongly positive if and only if it is ample. The *gonality* $\mathrm{gon}(X)$ of X is the minimal integer d such that there is a strongly positive and spanned sheaf F on X pure rank 1, depth 1, and $\deg(F) = d$. The *free gonality* f-gon(X) of X is the minimal integer d such that there are $L \in \mathrm{Pic}(X)$ and a strongly positive and spanned sheaf F on X with pure rank 1, depth 1, $\deg(L) = d$, and $F \subseteq L$. The *map gonality* m-gon(X) of X is the mimimal integer d such that there is an ample and spanned $L \in \mathrm{Pic}(X)$ such that $\deg(L) = d$. Obviously, $\mathrm{gon}(X) \leq \text{f-gon}(X) \leq \text{m-gon}(X)$. Easy examples show that these inequalities are in general strict (even if the curve is integral and nodal). In several examples we will see that we may refine these invariants restricting either to sheaves with $h^1 = 0$ (non-special sheaves) or to sheaves with $h^1 \neq 0$ (special sheaves). For irreducible curves of genus ≥ 2 only special sheaves compute the gonality, but in general this is not true for stable curves (take a graph curve in the sense of [2] having dual graph $\|X\|$ not 2-connected and apply [2], Proposition 2.5). For the free gonality we may also distinguish the cases $h^1(X, F) > 0$ and $h^1(X, L) = 0$. We need the following definition due to F. Catanese (see [4]). We change the terminology, because our set-up is simpler: any point of X contained in at least two components of X is an ordinary node of X. We recall that a reduced and connected projective curve with only planar singularities has arithmetic genus zero if and only if it is nodal, its associated dual graph is contractible and its irreducible components are smooth and rational (see [4], Proposition 1.8). From now on we assume $g > 0$. Let A be a connected union of some of the irreducible components of X. We say that A is a *rational tail* of X (a negative tail in the sense of [4], Definition 1.15) if it is connected, $p_a(A) = 0$, $\sharp(A \cap A^{[c]}) = 1$, and it is maximal with respect to these two properties. The maximality condition implies that any two rational tails of X are disjoint. Let X_- denote the union of all $T \in \mathcal{B}(X)$ not contained in a rational tail. Since $g > 0$, $X_- \neq \emptyset$. It is easy to check that X_- is connected and that it has no rational tail. Obviously, $p_a(X_-) = p_a(X)$. We say that X_- is the *rational reduction* of X.

Here is a sample of our results.

Theorem 1. *Assume that the dual graph $\|X\|$ is contractible. Then $\mathrm{gon}(X) = \sum_{T\in\mathcal{B}(X)} \mathrm{gon}(T)$, f-gon$(X) = \sum_{T\in\mathcal{B}(X)}$ f-gon(T), and m-gon$(X) = \sum_{T\in\mathcal{B}(X)}$ m-gon(T). Moreover, if L computes the m-gonality of X, then $h^0(X, L) = 1+\sharp(\mathcal{B}(X))$. Let G be any strongly positive and spanned sheaf with pure rank 1 and depth 1 appearing in the definition of f-gonality. Then $h^0(X, G) = s+1$ and G is locally free at each point of X lying on two different irreducible com-*

ponents.

Theorem 2. *Let X_i, $i = 1, 2$, be integral projective curves. Set a_i, $i = 1, 2$, be the gonality (resp. f-gonality, resp. m-gonality) of X_i and F_i a sheaf on X_i computing it. Let $f_i : C_i \to X_i$ be the normalization map. Set $R_i := f_i^*(F_i)/\mathrm{Tors}(f_i^*(F_i))$ and $b_i = \deg(R_i)$ ($b_i = a_i$ for the m-gonality, because in this case F_i is locally free). Fix an integer t such that $1 \le t \le \min\{b_1, b_2\}$. Then there exists a connected and reduced projective curve X with the following properties:*

(a) X has exactly two irreducible components, say Y and C, and $Y \cong X_1$, $C \cong X_2$;

(b) $\sharp(Y \cap C) = t$ and every point of $Y \cap C$ is an ordinary node of X;

(c) $\mathrm{gon}(X) = a_1 + a_2$ (resp. f-$\mathrm{gon}(X) = a_1 + a_2$, resp. m-$\mathrm{gon}(X) = a_1 + a_2$).

Proposition 1. *Let Y be a reduced projective curve. Fix any ordinary node of Y and let $u : D \to Y$ be the partial normalization of Y in which we normalize only the point P. Assume that D is connected. Then $\mathrm{gon}(D) \le \mathrm{gon}(Y) \le \mathrm{gon}(D) + 1$. If $\mathrm{gon}(Y) = \mathrm{gon}(D)$, then every sheaf computing $\mathrm{gon}(Y)$ is locally free at P.*

Corollary 1. *Assume $s := \sharp(\mathcal{B}(X)) \ge 2$ and let t be the number of the singular points of X lying on at least two irreducible components of X. Assume that each of these t points is an ordinary node of X. Then $\sum_{T \in \mathcal{B}(X)} \le \mathrm{gon}(T) \le \mathrm{gon}(X) \le \sum_{T \in \mathcal{B}(X)} + t + 1 - s$.*

We raise the following two questions.

Question 1. Is it possible to characterize all X such that there is a strongly positive $L \in \mathrm{Pic}(X)$ with $h^1(X, L) = 0$ and $\deg(L) = \max\{\sharp(\mathcal{B}(X), g - 1\}$?

See Corollary 2 for some X such that there is a strongly positive $L \in \mathrm{Pic}(X)$ with $h^1(X, L) = 0$ and $\deg(L) = \max\{\sharp(\mathcal{B}(X), g - 1\}$.

Question 2. Assume the existence of a strongly positive $L \in \mathrm{Pic}(X)$ such that $\deg(L) = g - 1$ and $h^1(X, L) = 0$. For any $P \in X_{reg}$ we have $h^1(X, L(P)) = 0$ and $h^0(X, L(P)) = 1$ (Lemma 2). Under what assumptions on X, L, and P is the base locus of $L(P)$ a finite set? For any sheaf F with pure rank 1 and depth 1 such that $L \subset F$ and $\deg(F) = g$ we have $h^0(X, F) = 1$ (Lemma 2). Under what assumptions on X, L, and F, is the base locus of F a finite set?

For recent work on the Brill-Noether theory of stable curves, see [3].

2. Proofs and Related Results

Remark 1. Let F be a sheaf on X with pure rank 1 and depth 1. Assume that F is locally free at all singular points of X which are neither ordinary nodes nor ordinary cusps. Set $t := \deg(X)$ and $a := \sharp(\mathrm{Sing}(X))$. By the classification of pure rank 1 modules with depth 1 on a ordinary node (see [6], pp. 164–166) or a simple singularity (see [5]) we see that there is $L \in \mathrm{Pic}(X)$ containing F and with degree d if and only if $d \geq t + a$.

Remark 2. For every $T \in \mathcal{B}(X)$ fix $L^T \in \mathrm{Pic}(T)$. Since $\dim(X) = 1$, there is a finite open covering $\{U_i\}_{i \in I}$ of X such that $L^T|T \cap U_i \cong \mathcal{O}_{T \cap U_i}$ for all $T \in \mathcal{B}(X)$ and all $i \in I$. Hence a suitable trivialization shows the existence of $L \in \mathrm{Pic}(X)$ such that $L|T \cong L^T$ for all $T \in \mathcal{B}(X)$. Thus the restriction map $\rho : \mathrm{Pic}(X) \to \times_{T \in \mathcal{B}(X)} \mathrm{Pic}(T)$ is surjective. For every $T \in \mathcal{B}(X)$ fix a sheaf F^T on T with pure rank 1 and depth 1 such that F^T is locally free at each point of $T \cap T^{[c]}$. An open covering as above shows the existence of a sheaf F on X with pure rank 1 and depth 1 such that $F|T \cong F^T$ for all $T \in \mathcal{B}(X)$ and F is locally free around any point lying on at least 2 irreducible components of X.

Example 1. Set $s := \sharp(\mathcal{B}(X))$. Notice that $p_a(X) = 0$ if and only if X is nodal, $T \cong \mathbb{P}^1$ for all $T \in \mathcal{B}(X)$ and $\|X\|$ is contractible, i.e. there is an ordering $T_1, \dots, T_s$ of $\mathcal{B}(X)$ such that $T_1 \cup \cdots \cup T_i$ is connected and $\sharp((T_1 \cup \cdots \cup T_i) \cup T_{i+1}) = 1$ for all $i \in \{1, \dots, s-1\}$ (see [4], Proposition 1.8). Thus the group homomorphism $\alpha : \mathrm{Pic}(X) \to \mathbb{Z}^{\oplus s}$ defined by the formula $\alpha(L) := (\deg(L|T_1), \dots, \deg(L|T_s))$ is an isomorphism. Using $s-1$ Mayer-Vietoris exact sequence (use 1) below), we easily see that $L \in \mathrm{Pic}(X)$ is spanned if and only if $\deg(L|T_i) \geq 0$ for all $i \in \{1, \dots, s\}$ and that if L is spanned, then $h^1(X, L) = 0$. Hence any strongly positive line bundle on X is spanned and non-special. There is a strongly positive degree d line bundle on X if and only if $d \geq s$. Notice that $\sharp(\mathrm{Sing}(X)) = s - 1$. There is a strongly positive purely rank 1 sheaf on X such that $\mathrm{depth}(F) = 1$, $\deg(F) = d$ and $\sharp(\mathrm{Sing}(F)) = a$ if and only if $0 \leq a \leq s-1$ and $d \geq a + s$.

Remark 3. The map-gonality m-gon(X) of X is the minimal integer d such that there is a morphism $f : X \to \mathbb{P}^r$ (for some $r \geq 1$), f is finite (i.e. it contracts no irreducible component of X) and $\deg(f) \cdot \deg(f(X)) = d$. If $r \geq 2$, then taking a general projection into $\mathbb{P}^1$ we get a morphism as above with $r = 1$. However, in general it will not be linearly normal. Here is a stupid example. Fix an integer $d \geq 2$ and let $X \subset \mathbb{P}^2$ a union of d general lines. Then gon(X) = f-gon(X) = m-gon$(X) = d$, but any morphism computing m-gon(X) is either an embedding $X \hookrightarrow \mathbb{P}^2$ as a degree d plane curve or a linear projection of such an embedding.

Remark 4. Let T be an integral projective curve with only planar singularities. Set $q := p_a(T)$. Since T has smoothable singularities, there exist an affine, connected curve U, $o \in U$, and a flat family $\{Y_t\}_{t\in U}$ such that $Y_o \cong T$ and Y_t is a smooth and connected genus q curve. Classical Brill-Noether theory gives $\mathrm{gon}(Y_t) \le \lfloor (q+3)/2 \rfloor$ for all $t \in U\backslash\{o\}$. Hence $\mathrm{gon}(T) \le \lfloor (q+3)/2 \rfloor$.

Lemma 1. *The following inequalities are true:* $\mathrm{gon}(X) \ge \sum_{T\in\mathcal{B}(X)} \mathrm{gon}(T)$, $\text{m-gon}(X) \ge \sum_{T\in\mathcal{B}(X)} \text{m-gon}(T)$ *and* $\text{f-gon}(X) \ge \sum_{T\in\mathcal{B}(X)} \text{f-gon}(T)$*. Moreover, if F is a pure rank 1 strongly positive and spanned depth 1 sheaf on X computing $\mathrm{gon}(X)$ and b is the number of points of X lying on two irreducible components of X and at which F is not locally free, then* $\mathrm{gon}(X) \ge b + \sum_{T\in\mathcal{B}(X)} \mathrm{gon}(T)$.

Proof. Fix $L \in \mathrm{Pic}(X)$ computing $\text{m-gon}(X)$. Thus for each $T \in \mathcal{B}(X)$ the line bundle $L|T$ is spanned and with positive degree. Thus $\deg(L|T) \ge \text{m-gon}(T)$. Since L is locally free, $\deg(L) = \sum_{T\in\mathcal{B}(X)} \deg(L|T)$. Hence we get the second inequality. Take $A \in \mathrm{Pic}(X)$ computing $\text{f-gon}(X)$ and a pure rank 1 strongly positive and spanned subsheaf B of A. Since A is a non-zero subsheaf of A, it has depth 1. Fix any $T \in \mathcal{B}(X)$. Since B is strongly positive, $\deg((B|T)/\mathrm{Tors}(B|T)) > 0$. Since B is spanned, $(B|T)/\mathrm{Tors}(B|T)$ is spanned. Since $R|T \in \mathrm{Pic}(T)$ and $(B|T)/\mathrm{Tors}(B|T)$ is a subsheaf of $R|T$, $\deg(R|T) \ge \text{f-gon}(T)$. Since R is locally free, $\deg(R) = \sum_{T\in\mathcal{B}(X)} \deg(R|T)$. Hence we get the third inequality. Let F is a pure rank 1 strongly positive and spanned depth 1 sheaf on X computing $\mathrm{gon}(X)$. Let $S \subset X$ be the set of all points of X lying on two irreducible components of X and at which F is not locally free. Set $b := \sharp(S)$. If $b = 0$, then we may use the previous proof, because $F|T$ has depth 1 for all $T \in \mathcal{B}(X)$ and $\deg(F) = \sum_{T\in\mathcal{B}(X)} \deg(F|T)$. Hence we may assume $b > 0$. Fix any $Q \in S$ and let $f_Q : X_Q \to X$ be the partial normalization of X in which we normalizes only Q. Hence $f_Q^{-1}(Q)$ is the union of two smooth points Q', Q'' of X_Q. The germ F_Q of F at Q is isomorphic to the maximal ideal of of the local ring $\mathcal{O}_{X,Q}$. Hence the torsion part of $f_Q^*(G)$ has length 2, being given by two one-dimensional skyscraper sheaves, one supported by Q' and another one supported by Q''. Moreover, $\deg(f_Q^*(F)/\mathrm{Tors}(f_Q^*(F))) = \deg(F) - 1$. Let $f_S : X_S \to X$ be the partial normalization of X in which we normalize only the points of F. As above we get $\deg(f_S^*(F)/\mathrm{Tors}(f_S^*(F))) = \deg(F) - b$. Notice that each of the inequality above is true for disconnected curves, if it is true for all connected curves. Since $f_S^*(F)/\mathrm{Tors}(f_S^*(F))$ has no singular point lying on two irreducible components of X_S, the case $b = 0$ gives $\deg(f_S^*(F)/\mathrm{Tors}(f_S^*(F))) \ge \sum_{T\in\mathcal{B}(X)} \mathrm{gon}(T)$ even if

X_S is not connected, concluding the proof of the first inequality and of the last assertion of the lemma. □

Proposition 2. *Let Y be a reduced and connected curve. Assume the existence of connected subcurves $Y_i \subset Y$, $1 \le i \le s$, such that $\sharp(Y_1 \cup \cdots \cup Y_i) \cap Y_{i+1}) = 1$ for all $1 \le i \le s-1$, and that all points $Y_1 \cup \cdots \cup Y_i) \cap Y_{i+1}$, $1 \le i \le s-1$, are ordinary nodes of Y. Then $gon(Y) = \sum_{i=1}^{s} gon(Y_i)$, $f\text{-}gon(Y) = \sum_{i=1}^{s} f\text{-}gon(Y_i)$, and $m\text{-}gon(Y) = \sum_{i=1}^{s} m\text{-}gon(Y_i)$.*

Proof. The proof of Lemma 1 gives the inequalities $\mathrm{gon}(Y) \ge \sum_{i=1}^{s} \mathrm{gon}(Y_i)$, $\text{f-gon}(Y) \ge \sum_{i=1}^{s} \text{f-gon}(Y_i)$, and $\text{m-gon}(Y) \ge \sum_{i=1}^{s} \text{m-gon}(Y_i)$. Hence it is sufficient to prove the reverse inequalities. By induction on s we see that it is sufficient to do the case $s = 2$. Set $\{P\} := Y_1 \cap Y_2$. Since P is an ordinary node of X, P is a smooth point of both Y_1 and Y_2. Hence each depth 1 sheaf on Y_i is locally free at P_i. Since P is an ordinary node of Y, Y_1 and Y_2 are transversal at P, i.e. P is the scheme-theoretic intersection of Y_1 and Y_2 in Y. Hence for any depth 1 sheaf F on Y which is locally free at P the following Mayer-Vietoris type sequence is exact

$$0 \to F \to F|Y_1 \oplus F|Y_2 \to F|\{P\} \to 0\,. \tag{1}$$

Moreover, for all sheaves F_i on Y_i with rank 1 and depth 1 there is a unique depth 1 sheaf F such that $F|Y_1 \cong F_1$ and $F|Y_2 \cong F_2$; indeeed, if $F|Y_1$ has depth 1, F must be locally free at P. Hence (1) gives $\text{f-gon}(Y) \le \text{f-gon}(Y_1)+\text{f-gon}(Y_2)$ and $\text{m-gon}(Y) \le \text{m-gon}(Y_1)+\text{m-gon}(Y_2)$. Since any depth 1 sheaf on Y_i is locally free at P, (1) also gives the inequality $\mathrm{gon}(Y) \le \mathrm{gon}(Y_1) + \mathrm{gon}(Y_2)$. □

Lemma 2. *Let F, G be sheaves on X with pure rank 1 and depth 1 such that $F \subseteq G$ and $h^1(X, F) = 0$. Then $h^1(X, G) = 0$ and $h^0(X, G) = h^0(X, F) + length(G/F)$. If F is spanned, then G is spanned. If the base locus of F is finite, then the base locus of G is finite.*

Proof. Look at the exact sequence on X:

$$0 \to F \to G \to G/F \to 0\,. \tag{2}$$

Since G/F has finite support, $h^1(X, G/F) = 0$. Since $h^1(X, F) = 0$, (2) gives $h^1(X, G) = 0$ and $h^0(X, G) = h^0(X, F) + \mathrm{length}(G/F)$. Now assume that F is spanned. Let G' be the subsheaf of G spanned by $H^0(X, G)$. Thus G' is spanned and $h^0(X, G') = h^0(X, G)$. Since F is spanned, G' contains F. Hence the first part of the lemma applied to the inclusion $F \hookrightarrow G'$ gives $h^1(X, G') = 0$ and $h^0(X, G') = h^0(X, F) + \mathrm{length}(G'/F)$. Since $h^0(X, G') = h^0(X, G)$ and

G/G' has finite support, we get $G' = G$, proving the spannedness of G. In the same way we get the last assertion. □

Remark 5. Fix integers $d > t > 0$. Assume the existence of a strongly positive and spanned sheaf F on X with pure rank 1 and depth 1 such that $h^1(X, F) = 0$ and $\deg(F) = t$. Lemma 2 gives the existence of a strongly positive and spanned sheaf G on X with pure rank 1 and depth 1 such that $h^1(X, G) = 0$ and $\deg(G) = d$. If F is locally free, then we may take G locally free. If every singular point of X is an ordinary node or an ordinary cusp, then to find such a sheaf G locally free it is necessary and sufficient to assume $d - t \geq \sharp(\mathrm{Sing}(F))$ (Remark 1).

Remark 6. Fix $L \in \mathrm{Pic}(X)$ such that $h^1(X, L)$. Let $A \subsetneq X$ be any union of some of the irreducible components of X. Since $A \cap A^{[c]}$ is finite, $h^1(A \cap A^{[c]}, L|A \cap A^{[c]}) = 0$. Thus the Mayer-Vietoris exact sequence

$$0 \to L \to L|A \oplus L|A^{[c]} \to L|(A \cap A^{[c]}) \to 0 \tag{3}$$

gives $h^1(A, L|A) = h^1(A^{[c]}, L|A^{[c]}) = 0$.

Lemma 3. *Assume $\sharp(\mathcal{B}(A)) \leq p_a(A) - 1$ for all connected unions $A \subseteq X$ of some of the irreducible components of X. Then there exists a strongly positive line bundle L on X such that $h^1(X, L) = 0$ and $\deg(L) = g - 1$.*

Proof. The result is obvious if X is irreducible. Notice that the assumption on the connected subcurves is true for every proper subcurve B of X, because a connected subcurve of B is a connected subcurve of X. Hence we may assume $s := \sharp(\mathcal{B}(X)) \geq 2$ and that the result is true for all connected subcurves of X with at most $s - 1$ irreducible components. Since X is connected, there is $C \in \mathcal{B}(X)$ such that $C^{(c)}$ is connected. We fix $C \in \mathcal{B}(X)$ such that $Y := C^{(c)}$ is connected and for which $p_A(Y)$ is as large as possible. Since all points of $Y \cap C$ are ordinary nodes of X, $g = p_a(Y) + p_a(C) + \sharp(C \cap Y) - 1$.

Since Y is a connected subcurve of X, $s - 1 \leq p_a(Y) - 1$. By the inductive assumption there exists a strongly positive $R \in \mathrm{Pic}(Y)$ such that $\deg(R) = p_a(Y) - 1$ and $h^1(Y, R) = 0$. Riemann-Roch gives $h^0(Y, R) = 0$. We have $g > p_a(Y)$, unless $\sharp(Y \cap C) = 1$ and $C \cong \mathbb{P}^1$. Assume $\sharp(Y \cap C) = 1$ and $C \cong \mathbb{P}^1$. There is a unique $L \in \mathrm{Pic}(X)$ such that $L|Y \cong R$ and $L|C \cong \mathcal{O}_C$. Consider the Mayer-Vietoris exact sequence

$$0 \to L \to L|Y \oplus L|C \to L|(Y \cap C) \to 0\,. \tag{4}$$

Since $Y \cap C$ is scheme-theoretically a point, $L|C$ is spanned and $h^1(Y, L|Y) = h^1(C, L|C) = 0$, (4) gives $h^1(X, L) = 0$, proving the lemmas in this case. Hence we may assume $g > p_a(Y)$. Set $m := p_a(Y)$, $q := p_a(C)$ and $b := \sharp(Y \cap C)$. By the case $s = 1$ there is $N \in \mathrm{Pic}^{q-1}(C)$ such that $h^1(C, N) = 0$. Set $M := N(Y \cap C)$. Since every point of X on at least two irreducible components is an ordinary node of X, $C \cap Y \subset C_{reg}$. Since $Y \cap C$ is a degree b effective Cartier divisor of C, M is a degree $q+m-1$ line bundle on C. Let L be any line bundle on X such that $L|Y \cong R$ and $L|C \cong M$. Since $\deg(L) = \deg(R) + \deg(M) = g-1$, $\chi(L) = 0$. Hence $h^1(X, L) = 0$ if and only if $h^0(X, L) = 0$ (Riemann-Roch). Fix $\sigma \in H^0(X, L)$. Since $h^0(Y, R) = 0$, $\sigma|Y = 0$. Hence σ induces a section of $H^0(C, M)$ vanishing on $Y \cap C$. σ induces a section of N. Since $h^0(C, N) = 0$, (4) gives $\sigma = 0$, proving the "if" part of the lemma. □

Corollary 2. *Fix an integer $d \geq g-1$. Assume $\sharp(\mathcal{B}(A)) \leq p_a(A) - 1$ for all connected subcurves $A \subseteq X$. Then there exists a strongly positive line bundle L on X such that $h^1(X, L) = 0$ and $\deg(L) = d$.*

Proof. Lemma 2 and Remark 5 show that it is sufficient to check the case $d = g-1$, i.e. the case proved in Lemma 3. □

Remark 7. Assume the existence of a strongly positive $L \in \mathrm{Pic}(X)$ such that $\deg(L) = g-1$ and $h^1(X, L) = 0$. Since L is strongly positive, we have $\sharp(\mathcal{B}(X)) \leq g-1$. Hence the part $A = X$ of the assumption of Lemma 3 is a necessary condition. However, the same condition for all connected subcurves is not a necessary condition. For instance, it excludes the existence of $T \in \mathcal{B}(X)$ such that $p_a(T) \leq 1$.

Example 2. Here we show the existence of some nodal curve X such that there is no strongly positive $L \in \mathrm{Pic}(X)$ with $h^1(X, L) = 0$ and $\deg(L) = \max\{\sharp(\mathcal{B}(X), g-1\}$. Assume $g \geq 6$. Let Y be a graph curve of genus $g-3$ (see [2]). Fix $P \in Y_{reg}$. Let C be a smooth genus 3 curve. Fix $P' \in C$. Let X be the genus g nodal curve obtained gluing together Y at P with C at P'. Hence $p_a(X) = g$ and $\sharp(\mathcal{B}(X)) = 2(g-3)-2+1 = 2g-5$. Since $g \geq 6$, $\sharp(\mathcal{B}(X)) =\geq g$. Let L be a strongly posive line bundle on X with degree $\sharp(\mathcal{B}(X))$. Notice that $\deg(L|T) = 1$ for all $T \in \mathcal{B}(X)$. Hence $\deg(L|C) = 1$. Since $p_a(C) = 3$, Riemann-Roch gives $h^1(C, L|C) > 0$. Hence $h^1(X, L) > 0$ (Remark 6).

To check several examples we found useful the following improvement of Lemma 3

Lemma 4. *Fix an integer $d \geq g-1$. Assume the existence of a pair (B, N), where B is a connected subcurve of X, $N \in Pic(B)$, $\deg(N) = p_a(B) - 1$ and $h^1(B, N) = 0$. Assume $\sharp(\mathcal{B}(A)) \leq p_a(A) - 1$ for all connected subcurves $A \subseteq X$*

such that $A \subset B$. *Then there exists a strongly positive line bundle* L *on* X *such that* $h^1(X, L) = 0$ *and* $\deg(L) = d$.

Proof. In the proof of Lemma 3 take C with the additional restriction that $B \subseteq C^{[c]}$. The proof of Lemma 3 gives the case $d = g - 1$. Then apply Lemma 2 and Remark 5 to get the case $d > g - 1$. □

Lemma 5. *Assume* $g > 0$ *and the existence of a connected subcurve* $Y \subsetneq X$ *with the following properties. Set* $a := \sharp(\mathcal{B}(X) \backslash \mathcal{B}(Y))$ *and* $Y_0 := Y$. *Assume the existence of an ordering* $T_1, \ldots, T_a$ *of* $\mathcal{B}(X) \backslash \mathcal{B}(Y)$ *such that* $p_a(Y_{i+1}) \leq p_a(Y_i) + 1$ *for all* $0 \leq i \leq a - 1$, *where* $Y_j := Y_0 \cup Y_1 \cup \cdots \cup Y_j$ *for all* $j \in \{1, \ldots, a\}$, *but we have a strict inequality for at least one* $i \in \{1, \ldots, a-1\}$. *Then there exists no strongly positive* $L \in Pic(X)$ *such that* $\deg(L) = g - 1$ *and* $h^1(X, L) = 0$.

Proof. Fix a strongly positive $L \in \mathrm{Pic}(X)$ such that $h^1(X, L) = 0$. Notice that $p_a(Y) \geq g - a + 1$. Remark 6 gives $h^1(Y, L|Y) = 0$. Thus $\deg(L|Y) \geq p_a(Y) - 1 \geq g - a$. Since L is strongly positive, $\deg(L) \geq \deg(L|Y) + a \geq g$. □

Proposition 3. *Assume* $g > 0$ *and* $X_- \neq X$. *Set* $a := \sharp(\mathcal{B}(X) \backslash \mathcal{B}(X_-))$. *Let* m *be the first integer such that there is* $R \in Pic(X_-)$ *with* $\deg(R) = m$ *and* $h^1(X_-, R) = 0$. *There is a degree* d *strongly positive and non-special line bundle on* X *if and only if* $d \geq m + a$.

Proof. The "only if" part follows from Remark 6 and the fact that $\deg(L) \geq \deg(L_0) + a$ for every strongly positive line bundle on X. For the "if" part use a Mayer-Vietoris exact sequences to get the case $d = m + a$, and then apply Remark 5. □

Remark 8. Assume the existence of $L \in \mathrm{Pic}(X)$ such that $\deg(L) = g - 1$ and $h^1(X, L) = 0$. Fix $P \in X_{reg}$. Let B denote the set-theoretic base locus of $L(P)$. Let C be the irreducible component of X containing P. Obviously, $D \cap C$ is finite. Fix $Q \in X$. If $Q \notin B$ and $Q \in T$, then $B \cap T$ is finite.

Example 3. Here we assume $\sharp(\mathcal{B}(A)) = 2$ and call Y, C the irreducible components of X. We assume $\sharp(Y \cap C) = 1$, say $\{P\} = Y \cap C$, and $q_1 := p_a(Y) \geq 1$, and $q_2 := p_a(C) \geq 2$. Thus $g = q_1 + q_2 \geq 3$. Fix a general $R \in \mathrm{Pic}^{q_1}(Y)$ and a general $M \in \mathrm{Pic}^{q_2 - 1}(C)$. Thus $h^0(Y, R) = 1$, P is not a base point of $|R|$, $h^1(Y, R) = 0$, and $h^0(C, M) = h^1(C, M) = 0$. There is a unique $L \in \mathrm{Pic}(X)$ such that $L|Y \cong M$ and $L|C \cong M$. Notice that $\deg(L) = g - 1$. Since $q_1 \geq 1$ and $q_2 - 1 \geq 1$, L is strongly positive. Since P is a not a base point of $|R|$, the usual Mayer-Vietoris exact sequence gives

$h^0(X,L)=0$. Since Since $\deg(L)=g-1$, Riemann-Roch gives $h^1(X,L)=0$. Hence $h^1(X,L(Q))=0$ and $h^0(X,L(Q))=1$ for all $Q\in X_{reg}$ (Lemma 2). If $Q\in Y_{reg}\backslash\{P\}$, then $L(Q)|C\cong M$ and hence C is contained in the base locus of $L(Q)$. Now assume that Q is general in C. The generality of the pair (M,Q) easily gives that P is not in the base locus of $M(Q)$. Hence in this case we see that Y is not contained in the base locus of $L(Q)$. Remark 8 shows that in this case the base locus of $L(Q)$ is finite. Now we assume either $q_1=0$ and $q_2\geq 2$. We take (R,P,L) as above. We have $h^i(X,L)=0$, $i=0,1$, but L is not strongly positive (as was expected, because $C=X_-$). Fix a general $Q\in C$. As above we get $h^0(X,L(Q))=1$ and that the base locus of $L(Q)$ is finite. $L(Q)$ is strongly positive.

Example 4. Here we assume $s:=\sharp(\mathcal{B}(X))\geq 2$ and that $\|X\|$ is contractible, i.e. there is an ordering $T_1,\dots,T_s$ of $\mathcal{B}(X)$ such that $T_1\cup\dots\cup T_i$ is connected and $\sharp((T_1\cup\dots\cup T_i)\cup T_{i+1})=1$ for all $i\in\{1,\dots,s-1\}$. Set $q_i:=p_a(X_i)$. We have $g=q_1+\dots+q_s$. Set $\alpha:=\sharp(\{i\in\{1,\dots,s\}:q_i=0\})$ and $\beta:=\sharp(\{i\in\{1,\dots,s\}:q_i=1\})$.

Claim. *(a) There is $L\in Pic(X)$ such that $h^1(X,L)=0$ and $\deg(L)=t$ if and only if $t\geq g-1$.*

(b) There is a strongly positive $L\in Pic(X)$ such that $h^1(X,L)=0$ and $\deg(L)=t$ if and only if $t\geq g-1+2\alpha+\beta$.

(c) There is a strongly positive $L\in Pic(X)$ such that $h^1(X,L)=0$, $\deg(L)=t$ and the base locus of L is finite if and only if $t\geq\max\{g,g-1+2\alpha+\beta\}$.

Proof of the Claim. The "only if" parts are true by Remark 6. The "if" parts are proven by induction on s as in Example 3. □

Let L be aspanned line bundle on X. L is said to be birationally very ample if there is a finite set $S\subset X$ such that the morphism $h_L:X\to\mathbb{P}^n$, $n:=h^0(X,L)-1$, is an embedding.

Example 5. Take the set-up of Example 3, i.e. the set up of Example 4 with $s=2$. Let Y,C be the ireducible components of X. Set $q_1:=p_a(Y)$, $q_2:=p_a(C)=g-q_1$.

First Claim. *There is a strongly positive $L\in Pic(X)$ such that $h^1(X,L)=0$ and $\deg(L)=t$ if and only if $t\geq g+2$.*

Proof of the First Claim. We will first check the "if" part. By Lemma 2 it is sufficient to do the case $t=g+2$. Fix a general $R\in\mathrm{Pic}^{q_1+1}(Y)$ and a general $R\in\mathrm{Pic}^{q_2+1}(Y)$. Hence $h^1(C,M)=h^1(Y,R)=0$ and M,R are spanned. Let L be the unique line bundle on X such that $L|Y\cong Y$ and $L|C\cong M$. Since $\deg(R)-1=q_1-1$ and R is general, $h^1(Y,R(-(C\cap Y))=0$.

Hence the Mayer-Vietoris exact sequence (2) gives that the restriction map $\rho_{L,C} : H^0(X, L) \to H^0(C, R)$ is surjective. Since R is spanned, the base locus of L is disjoint from C, concluding the proof of the "if" part. Now we check the "only if" part. If L is non-special and spanned, then $L|Y$ and $L|C$ must be non-special and spanned. If $q_1 > 0$ and $q_2 > 0$, then we get $\deg(L|Y) \geq q_1 + 1$ and $\deg(L|C) \geq q_2 + 1$ and hence $\deg(L) \geq g + 2$. If either $q_1 = 0$ or $q_2 > 0$, then we use the strong positivity of L to get $\deg(L|Y) \geq q_1 + 1$ and $\deg(L|C) \geq q_2 + 1$. Hence even in this case $\deg(L) \geq g + 2$. □

Second Claim. *If $t \geq g + 4$ there is a strongly positive, spanned, and birationally very ample $L \in Pic(X)$ such that $h^1(X, L) = 0$ and $\deg(L) = t$.*

Proof of the Second Claim. Fix integers $t_1 \geq q_1 + 2$ and $t_2 \geq q_2 + 2$ such that $t_1 + t_2 = t$. Fix a general $R \in \mathrm{Pic}^{t_1}(Y)$ and a general $R \in \mathrm{Pic}^{t_2}(Y)$. Hence $h^1(C, M) = h^1(Y, R) = 0$ and M, R are spanned and birationally very ample. As above we get that L is spanned and that the restriction maps $H^0(X, L) \to H^0(Y, R)$ and $H^0(X, L) \to H^0(C, M)$ are surjective. Hence $h_L|Y$ and $h_L|C$ are birational onto their images, i.e. generically unramified and generically injective. Since $h^0(X, L) > \max\{h^0(Y, R), h^0(C, M)\}$, the irreducible curves $h_L(Y)$ and $h_L(C)$ are different. Thus $h_L(Y) \cap h_L(C)$ is finite. Hence there is a finite $S \subset X$ such that $h_L|X \backslash S$ is an embedding. □

In the same way we get the following statement.

Third Claim. *If $t \geq g + 6$ there is a very ample $L \in Pic(X)$ such that $h^1(X, L) = 0$ and $\deg(L) = t$.*

Remark 9. We stress the "only if" part the Second Claim of Example 5: any non-special, strongly positive and complete morphism $h_L : X \to \mathbb{P}^n$ has $n \geq 2$.

Example 6. Here we assume $\sharp(\mathcal{B}(A)) = 2$ and call Y, C the irreducible components of X. We assume $q_1 := p_a(Y)$, $q_2 = p_a(C)$, and $\sharp(Y \cap C) = x$, say $\{P_1, \dots, P_x\} = Y \cap C$. Hence $g = q_1 + q_2 + x - 1$. By Example 3 we may assume $x \geq 2$. We assume $q_2 \geq q_1$. Set $a_i := \max\{q_i - 1, 1\}$, $i = 1, 2$.

(a) Fix $L \in \mathrm{Pic}(X)$ such that $h^1(X, L) = 0$. Riemann-Roch gives $\deg(L) \geq g - 1$. Remark 6 gives $\deg(L|Y) \geq q_1 - 1$ and $\deg(L|C) \geq q_2 - 1$. L is strongly positive if and only if $\deg(L|Y) \geq 1$ and $\deg(L|Y) \geq 1$. If $h^0(X, L) > 0$, then Riemann-Roch gives $\deg(L) \geq g$.

First Claim. *There is a strongly positive $L \in Pic(X)$ such that $h^1(X, L) = 0$ and $\deg(L) = t$ if and only if $t \geq \max\{g - 1, a_1 + a_2\}$.*

Proof of the First Claim. Since $g = q_1 + q_2 + x - 1$, the "only if" part follows from (a). By Lemma 2 to check the "if" part in all cases it is sufficient to check the corresponding case with the equality for t instead of the inequality

$\geq$. Set $z := \max\{g-1, a_1+a_2\}$. Fix a general $R \in \mathrm{Pic}^{a_1}(Y)$ and a general $R \in \mathrm{Pic}^{z-a_1}(Y)$. Hence $h^1(C,M) = h^1(Y,R) = 0$. Let L be any line bundle on X such that $L|C \cong M$ and $L|Y \cong R$. From the Mayer-Vietoris exact sequence (4) we see that $h^1(X,L) = 0$ if and only if the difference map $v : H^0(Y, L|Y) \oplus H^0(C, L|C) \to H^0(Y \cap C, L|(Y \cap C))$ is surjective. Of course, in general the difference map v depends on L, not just R, M. First assume $a_1 = q_1 - 1$, i.e. $q_1 \geq 2$. In this case $H^0(Y,R) = 0$. Hence v is surjective if and only if the restriction map $v' : H^0(C,M) \to H^0(C \cap Y, M|C \cap Y)$ is surjective. Since $h^1(C,R) = 0$, v' is surjective if and only if $h^1(C, M(-(C \cap Y)) = 0$. Since $q_2 \geq q_1$, we also get $a_2 = q_2 - 1$. Thus $z = g-1 = q_1+q_2+x-1$. The degree x effective divisor $C \cap Y$ is a fixed divisor of C. Taking M general, we get $h^1(C, M(-(C \cap Y)) = 0$, because $\deg(M) - x = q_2 - 1$. If $q_1 \leq 1$, then we use a similar proof (here $a_1 = 1$). □

Second Claim. *There is a strongly positive $L \in Pic(X)$ such that $h^1(X,L) = 0$, $\deg(L) = t$ and the base locus of L is finite if and only if $t \geq \max\{g, a_1+a_2\}$.*

Proof of the Second Claim. The "if" part is obvious by Riemann-Roch (necessity of the inequality $t \geq g$) and the First Claim (necessity of the inequality $t \geq a_1 + a_2$). Now we check the "only if" part. Set $z := \max\{g, a_1+a_2\}$. By Lemma 2 it is sufficient to do the case $t = z$. First assume $q_1 \geq 1$. Fix a general $R \in \mathrm{Pic}^{q_1}(Y)$ and a general $M \in \mathrm{Pic}^{z-q_1}(C)$. Let L be any line bundle on X such that $L|C \cong M$ and $L|Y \cong R$. The proof of the First Claim gives $h^1(X,L) = 0$. The same proof gives the surjectivity of the restriction map $v' : H^0(C,M) \to H^0(C \cap Y, M|C \cap Y)$. From the exact sequence (4) we get that the surjectivity of v' implies the surjectivity of the restriction map $\rho_{L,Y} : H^0(X,L) \to H^0(Y, L|L)$. Since $\deg(R) = q_1$, the base locus of R is finite. Hence the base locus of L intersects Y in a finite set. Assume that the base locus of L contains C. Hence any section of L induces a section of R vanishing on $Y \cap C$. Since $Y \cap C$ is fixed, and $\deg(R) - x = q_1 - x \leq q_1 - 1$, we get a contradiction for general R to the inequality $h^0(Y,L) > 0$. Now assume $q_1 = 0$. Hence $a_1 = 1$. Fix $R \in \mathrm{Pic}^1(Y)$ and a general $M \in \mathrm{Pic}^{z-1}(C)$. Since $z - 1 \geq q_2 - 1 - x$, v' and $\rho_{L,Y}$ are surjective. Since R is very ample, in this case the base locus of L is disjoint from Y. Hence it is finite. □

Third Claim. *If $t \geq g+2$ there is a strongly positive $L \in Pic(X)$ such that $h^1(X,L) = 0$, $\deg(L) = t$ and the base locus of L is finite.*

Proof of the Third Claim. By Lemma 2 it is sufficient to do the case $t = g+2$. Fix a general $R \in \mathrm{Pic}^{q_1+1}(Y)$ and a general $M \in \mathrm{Pic}^{g+1-q_1}(C)$. Let L be any line bundle on X such that $L|C \cong M$ and $L|Y \cong R$. The proof of the First Claim gives $h^1(X,L) = 0$. The same proof gives the surjectivity of the restriction map $v' : H^0(C,M) \to H^0(C \cap Y, M|C \cap Y)$. From the exact sequence (4) we get that the surjectivity of v' implies the surjectivity of the restriction map $\rho_{L,Y} : H^0(X,L) \to H^0(Y,L|Y)$. Since $\deg(R) = q_1 + 1$ and R is general, R is spanned. Hence the base locus of L is disjoint from Y. □

Proof of Theorem 1. Lemma 1 gives $\mathrm{gon}(X) \geq \sum_{T\in\mathcal{B}(X)} \mathrm{gon}(T)$, $\text{m-gon}(X) \geq \sum_{T\in\mathcal{B}(X)} \text{m-gon}(T)$ and $\text{f-gon}(X) \geq \sum_{T\in\mathcal{B}(X)} \text{f-gon}(T)$. Hence it is sufficient to check the opposite inequalities. Since the proposition is obvious if X is irreducible, we may assume $s \geq 2$ and use induction on the number of the irreducible components. Since $\|X\|$ is contractible, there is $C \in \mathcal{B}(X)$ such that the associated vertex v_C lies only on one edge of $\|X\|$, i.e. $\sharp(C \cap Y) = 1$, where $Y := C^{[c]}$. Hence Y is connected. Our blanket assumption on X gives that the unique point P of $C \cap Y$ is an ordinary node, i.e. that $\{P\}$ is the scheme-theoretic intersection $C \cap Y$ and both C and Y are smooth at P. By the inductive assumption $\text{f-gon}(Y) = \sum_{T\in\mathcal{B}(Y)} \text{f-gon}(T)$, and $\text{m-gon}(Y) = \sum_{T\in\mathcal{B}(Y)} \text{m-gon}(T)$. Fix $R \in \mathrm{Pic}(Y)$ computing $\text{m-gon}(Y)$, $M \in \mathrm{Pic}(C)$ computing $\text{m-gon}(C)$, $R' \in \mathrm{Pic}(Y)$ computing $\text{f-gon}(Y)$, $M' \in \mathrm{Pic}(C)$ computing $\text{f-gon}(C)$, a spanned and strongly positive sheaf B on Y with pure rank 1 and depth 1 contained in R' and a spanned and strongly positive sheaf D on C with pure rank 1 and depth 1 contained in M'. The inductive assumption gives $h^0(Y,R) = h^0(Y,B) = s$. Since C is irreducible, it is classical and easy that $h^0(C,M) = h^0(C,D) = 2$. Since P is an ordinary node of X and $Y \cap C = \{P\}$, there are uniquely determined (up to isomorphisms) line bundles L, L' on X such that $L|Y \cong Y$, $L|C \cong M$, $L'|Y \cong R'$ and $L'|C \cong M'$. Since $P \in Y_{reg}$, B is locally free at P. Since $P \in C_{reg}$, D is locally free at P. Hence as above there is a unique, up to isomorphisms, depth 1 sheaf F on X such that $F|Y = B$ and $F|C = D$. We stress that we have the previous equalities without taking the quotient by the torsion, because F is locally free at P. Thus F has pure rank 1 and $\deg(F) = \deg(B) + \deg(D)$. Since $\deg(D) > 0$, $\deg(M) > 0$, $\deg(M') > 0$ and R, R', B are strongly positive, L, L', F are strongly positive. Thus to check the opposite inequalities it is sufficient to prove that L and F are spanned. Look at the Mayer-Vietoris exact sequence (4). Since M is spanned at P, the restriction map $\rho_{M,C\cap Y} : H^0(C,M) \to H^0(C \cap Y, M|C \cap M)$ is surjective. Hence the cohomology exact sequence of (4) gives $h^0(X,L) = h^0(Y,R) + h^0(C,M) - 1$ and that the restriction map $\rho_{L,Y} : H^0(X,L) \to H^0(Y,L|L)$ is surjective. Since R is spanned, we get that L is spanned at each point of Y. Since R is spanned at P, the restriction map $\rho_{R,Y\cap C} : H^0(Y,R) \to H^0(Y \cap C, R|Y \cap C)$ is surjective. Hence we get as above that L is spanned at all points of C. Hence L is spanned. The same proof works for F, because F is locally free at $\{P\} = Y \cap C$ and hence there is a Mayer-Vietoris exact sequence (4) with F instead of L. We also proved that $h^0(X,L) = h^0(X,F) = s+1$ for some L, F. Now we will check that this is true for all L computing the m-gonality of X. Fix any $L \in \mathrm{Pic}(X)$ computing the m-gonality. Since $\text{m-gon}(X) = \sum_{T\in\mathcal{B}(X)} \text{m-gon}(T)$, we get that $L|Y$ computes the m-gonality of Y and $L|C$ computes the gonality of C. Hence $h^0(C, L|C) = 2$. The inductive assumption gives $h^0(Y, L|Y) = s$. Use (4) to get $h^0(X, L) = h^0(Y, L|Y)+1$. Let G be any strongly positive sheaf with pure rank 1 and depth 1 appearing in the definition of f-gonality. Let F be the spanned sheaf computing f-gonality of X and constructed above. Recall that F is locally free at each point of X lying on two different irreducible components of X. Recall that $\deg(F) = \sum_{T\in\mathcal{B}(X)} \text{f-gon}(T)$ and $\deg(F|T) = \text{f-gon}(T)$ for all $T \in \mathcal{B}(X)$. Let S be the set of all singular points of X lying on two irreducible components of X and at which G is not locally free. Set $b := \sharp(S)$ and assume $S \neq \emptyset$. Fix any $Q \in S$ and let $f_Q : X_Q \to X$ be the partial normalization of X in which we normalize only Q. Hence $f_Q^{-1}(Q)$ is the union of two smooth points Q', Q'' of X_Q. The germ G_Q of G at Q is isomorphic to the maximal ideal of of the local ring $\mathcal{O}_{X,Q}$. Hence the torsion part of $f_Q^*(G)$ has $h^0 = 2$, being given by two one-dimensional skyscraper sheaves, one supported by Q'' and another one supported by Q''. Moreover, $\deg((f_Q^*(G)/\mathrm{Tors}(f_Q^*(G)) = \deg(G) - 1$. Iterating for all points of S we get $\sum_{T\in\mathcal{B}(X)} \deg((G|T)/\mathrm{Tors}(G|T)) = \deg(G) - b$. Since $\deg(G) = \deg(F)$, $\deg(F) = \sum_{T\in\mathcal{B}(X)} \deg(F|T)$ and $\deg((G|T)/\mathrm{Tors}(G|T)) \geq \text{f-gon}(T) = \deg(F|T)$ for all T, the assumption $b > 0$ gives a contradiction. □

Proof of Theorem 2. By Lemma 1 we have $\mathrm{gon}(X') \geq a_1 + a_2$ (respectively the same inequality for f-gonality or m-gonality) for every connected curve X' union of two irreducible components isomorphic to X_1, X_2 and intersecting transversally at an arbitrary number of points. Hence it is sufficient to prove the reverse inequality for some X. We only write down the proof of the case $a_i := \mathrm{gon}(X_i)$, $i = 1, 2$, the other cases being similar. Since X_i is irreducible and F_i compute the gonality of X_i, $h^0(X_i, F_i) = 2$. Fix a general $P_i \in X_i$. R_i is a line bundle on C_i spanned by the image W_i in $H^0(C_i, R_i)$ of $f_i^*(H^0(X_i, F_i))$. Since R_i is ample, $\dim(W_i) \geq 2$. Since $h^0(X_i, F_i) = 2$, $\dim(W_i) \leq 2$. Hence $\dim(W_i) = 2$. The pair (R_i, W_i) induces a degree b_i morphism $h_i : C_i \to \mathbb{P}^1$. Fix a general $O_i \in \mathbb{P}^1$. Hence $\sharp(h_i^{-1}(O_i)) = b_i \geq t$. Fix any $S_i \subseteq h_i^{-1}(O_i)$ such that $\sharp(S_i) = t$. By the generality of O_i, $A_i := f_i(S_i)$ is a set of t distinct smooth points of X_i. Fix isomorphisms $j_1 : X_1 \to Y$, $j_2 : X_2 \to C$. Let X be the only reduced curve with Y, C as irreducible components, $Y \cap C = j_1(A_1) = j_2(A_2)$, and each point of $Y \cap C$ an ordinary node of X. Set $G_i := (j_1^{-1})^*(F_i)$. Since S_i is contained in the smooth locus of X_i and F_i has depth 1, F_i is locally free at each point of A_i. Thus G_i is locally free at each point of $Y \cap C$. Hence there is a purely rank 1 sheaf G on X with depth 1 such that $G|Y \cong G_1$ and $G|C \cong G_2$ (G is unique if and only if $t = 1$; if $t > 1$ the set of all such sheaves has dimension $t-1$). We fix any such G. Since G is locally free at each point of $Y \cap C$, $\deg(G) = \deg(G|Y) + \deg(G|C) = a_1 + a_2$. Hence to conclude the proof it is sufficient to show that G is spanned. Since each G_i is spanned, it is sufficient to prove that the restriction maps $\rho : H^0(X,G) \to H^0(Y,G_1)$ and $\rho' : H^0(X,G) \to H^0(C,G_2)$ are surjective. We check the surjectivity of ρ, since the surjectivity of ρ' require only notational modifications. Look at the Mayer-Vietroris exact sequence (4) with G instead of L. It is exact, because G is locally free at each point of $Y \cap C$. Fix $P \in Y \cap C$ and $\sigma \in H^0(Y,G_1)\backslash\{0\}$. Since G is locally free at P, we may see $\sigma(P)$ as an element of the one-dimensional $\mathbb{K}$-vector space $G|\{P\}$. Since G_2 is spanned, there is $\eta \in H^0(C,G_2)$ such that $\sigma(P) = \eta(P)$. Notice that $f_i(j_i^{-1}(P)) = O_i$, $i = 1,2$. Hence the choices of S_1 and S_2 give $\sigma|Y \cap C$ and $\eta|Y \cap C$. Hence the pair (σ,η) goes to zero by the difference map $H^0(Y,G|Y) \oplus H^0(C,G|C) \to H^0(Y \cap C, G|Y \cap C)$ of the long cohomology exact sequence of (4). Hence this cohomology exact sequence shows that the pair (σ,η) induces $\alpha \in H^0(X,G)$ such that $\rho(\alpha) = \sigma$. □

Proof of Proposition 1. Let F be a strongly positive and spanned depth 1 sheaf on D computing $\mathrm{gon}(D)$. Set $G := u_*(F)$. Obviously, G has no torsion, i.e. it has depth 1. Since u induces a bijection between $\mathcal{B}(D)$ and $\mathcal{B}(X)$, G has pure rank 1. Let G' be the subsheaf of G spanned by $H^0(X,G)$. Since $h^0(X,G) = h^0(D,F)$ and F is spanned, G is spanned outside P, i.e. the sheaf G/G' is supported by P. Since u is finite, $h^1(X,G) = h^1(D,F)$. Hence G' is depth 1 and with pure rank 1. Fix $T \in \mathcal{B}(D)$. Since F is strongly positive and spanned, the depth 1 sheaf $(F|T)/\mathrm{Tors}(F|T)$ has at least two sections. Hence the sheaf $(G'|u(T))/\mathrm{Tors}(G'|u(T))$ has at least two sections. Hence it has degree > 0. Thus G' is totally positive. Since $p_a(D) = p_a(Y)$ and $h^i(D,F) = h^i(D,G)$ for $i = 0,1$, Riemann-Roch gives $\deg(G) = \deg(F) + 1$. Hence $\deg(G') \leq \deg(F) + 1 = \mathrm{gon}(D) + 1$. Hence $\mathrm{gon}(X) \leq \mathrm{gon}(D) + 1$. Let A be a strongly positive and spanned pure rank 1 sheaf on X computing $\mathrm{gon}(X)$. Set $A' := u^*(A)/\mathrm{Tors}(u^*(A))$. A' is spanned, strongly positive and with pure rank 1. Hence $\deg(A') \geq \mathrm{gon}(D)$. If A is locally free at P, then $f^*(A)$ has depth 1 and $\deg(A') = \deg(A)$. If A is not locally free at P, then $\deg(A') = \deg(A) - 1$ (see the proof of Lemma 1). Hence we get the inequality $\mathrm{gon}(D) \leq \mathrm{gon}(X)$ and the last assertion. □

Proof of Corollary 1. There is a set S of $t+1-s$ singular points of X, each of them lying on two different irreducible components and such that $\|X_S\|$ is contractible, where $f_S : X_S \to X$ is the partial normalization of X in which we normalize only the points of S. Notice that f_S maps isomorphically each irreducible component of X_S onto a different irreducible component of X. Thus $\sum_{T\in\mathcal{B}(X_S)} = \sum_{T\in\mathcal{B}(X)}$. Proposition 1 gives $\mathrm{gon}(X_S) = \sum_{T\in\mathcal{B}(X_S)}$. Choose an ordering of the points of S to decompose f_S into a sequence of $t+1-s$ normalizations of a single node. Then apply $t+1-s$ times Proposition 1. $\square$

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References

[1] A. Altman, S. Kleiman, Introduction to Grothendieck duality theory, *Lect. Notes in Math.*, **146**, Springer, Berlin (1970).

[2] D. Bayer, D. Eisenbud, Graph curves. with an appendix by Sung Won Park, *Adv. Math.*, **86**, No. 1 (1991), 1-40.

[3] L. Caporaso, Brill-Noether theory of binary curves, *ArXiv:* math/0807.1484.

[4] F. Catanese, Pluricanonical Gorenstein curves, In: *Enumerative Geometry and Classical Algebraic Geometry*, Nice (1981), 51-95; In: *Progress in Math.*, **24**, Birkhäuser, Basel (1982).

[5] G.-M. Greuel, H. Knörrer, Einfache Kurvensingularitäten und torsionfreie Moduln, *Math. Ann.*, **270** (1985), 417-425.

[6] C. Seshadri, Fibrés vectoriels sur les courbes algébriques, *Astérisque*, **96** (1982).