International Journal of Pure and Applied Mathematics
Volume 50 No. 3 2009, 359-372

ON THE SECANT VARIETIES TO THE TANGENT DEVELOPABLE OF VERONESE VARIETIES

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $V_{n,d} \subseteq \mathbf{P}^N$, $N := \binom{n+d}{n} - 1$, be the order d Veronese embedding of $\mathbf{P}^n$, $X_{n,d} := T(V_{n,d}) \subseteq \mathbf{P}^N$ the tangent developable of $V_{n,d}$ and $S^{s-1}(X_{n,d}) \subseteq \mathbf{P}^N$ the s-secant variety of $X_{n,d}$, i.e. the closure in $\mathbf{P}^N$ of the union of all $(s-1)$-linear spaces spanned by s points of $X_{n,d}$. $S^{s-1}(X_{n,d})$ has expected dimension $\min\{N, (2n+1)s - 1\}$. Catalisano, Geramita and Gimigliano conjectured that $S^{s-1}(X_{n,d})$ has always the expected dimension, except when $d = 2$, $n \geq 2s$ or $d = 3$ and $n = 2, 3, 4$. In this paper we prove their conjecture when $n = 4$ and $n = 5$, $d \geq 4$, and an asympotic case of the conjecture for all $n \geq 6$.

AMS Subject Classification: 14N05
Key Words: tangent developable, secant variety, Veronese variety, fat point, zero-dimensional scheme, postulation

1. Introduction

Let $Y \subset \mathbf{P}^N$ be a closed and integral m-dimensional variety and t a non-negative integer. The t-secant variety $S^t(Y) \subset \mathbf{P}^N$ of Y is the closure in $\mathbf{P}^N$ of the union of all t-dimensional linear spaces spanned by $t+1$ distinct points of Y. Thus $S^t(Y)$ is irreducible, $S^0(Y) = Y$ and $\dim(S^t(Y)) \leq \min\{(t+1)(m+1) - 1, N\}$ (see [1]). We will say that Y is $(t+1)$-nondegenerate if $\dim(S^t(Y)) = \min\{(t+1)(m+1) - 1, N\}$. Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety. Let $T(X) \subset \mathbf{P}^N$ be the tangent developable of X, i.e. the closure in $\mathbf{P}^N$ of the union of all embedded tangent spaces $T_P X \subset \mathbf{P}^N$ with $P \in X_{reg}$. Thus $T(X)$ is irreducible and $\dim(T(X)) \leq \min\{N, 2n\}$. We will say that X is tangentially

Received: September 16, 2008

nondegenerate or tangentially 1-ordinary if $\dim(T(X)) = 2n$. For any positive integer s we will say that X is tangentially s-nondegenerate or tangentially s-ordinary if $\dim(T(X)) = 2n$ and $T(X)$ is s-ordinary, i.e. if $\dim(T(X)) = 2n$ and $\dim(S^s(T(X))) = \min\{(s+1)(2n+1)-1, N\}$.

Let $V_{n,d} \subseteq \mathbf{P}^N$, $N := \binom{n+d}{n} - 1$, be the order d Veronese embedding of $\mathbf{P}^n$ and $X_{n,d} := T(V_{n,d}) \subseteq \mathbf{P}^N$ the tangent developable of $V_{n,d}$. M.V. Catalisano, A.V. Geramita and A. Gimigliano conjectured that every secant variety of $X_{n,d}$ has the expected dimension, except when $d = 2$ and $n \geq 2s$ or $d = 3$ and $n = s = 2, 3, 4$ (see [6], Conjecture 4.1). They showed that in the exceptional cases $d = 2$ and $n \geq 2s$ or $d = 3$ and $n = s = 2, 3, 4$ the variety $S^{s-1}(X_{n,d})$ has not the expected dimension. In [4] we used the Horace method (see [8], [2], [3]) to prove this conjecture for $n = 2$ and $n = 3$. A key point of [6] was the reduction of their conjecture to a postulation problem for general unions of certain zero-dimensional schemes, called (2,3)-points. Fix a line $L \subseteq \mathbf{P}^n$ and $P \in L$. The (2,3)-point associated to (P, L) is the closed subscheme of $\mathbf{P}^n$ with $(\mathcal{I}_P)^3 + (\mathcal{I}_L)^2$ as its ideal sheaf. It has length $2n+1$ and it is intermediate between the double point $2P$ and the triple point $3P$. In [4] we proved the conjecture raised in [6], for $n = 2, 3$. Here we will use again the Horace method to prove the conjecture for $n = 4, 5$ (except the case $(n, d) = (5, 3)$) and an asymptotic form of it for all $n \geq 6$.

Theorem 1. *Fix integers d, s such that $d \geq 3$, $s > 0$ and $(d, s) \neq (3, 4)$. Let $Z \subset \mathbf{P}^4$ be a general union of s (2,3)-points. Then either $h^1(\mathbf{P}^4, \mathcal{I}_Z(d)) = 0$ (case $9s \leq \binom{d+4}{4}$) or $h^0(\mathbf{P}^4, \mathcal{I}_Z(d)) = 0$ (case $9s \geq \binom{d+4}{4}$). Equivalently, for all (d, s) as above the $(s-1)$-th secant variety $S^{s-1}(X_{4,d})$ of $X_{4,d}$ has the expected dimension.*

Theorem 2. *Fix integers d, s such that $d \geq 4$, $s > 0$. Let $Z \subset \mathbf{P}^5$ be a general union of s (2,3)-points. Then either $h^1(\mathbf{P}^5, \mathcal{I}_Z(d)) = 0$ (case $11s \leq \binom{d+5}{5}$) or $h^0(\mathbf{P}^5, \mathcal{I}_Z(d)) = 0$ (case $11s \geq \binom{d+5}{5}$). Equivalently, for all (d, s) as above the $(s-1)$-th secant variety $S^{s-1}(X_{5,d})$ of $X_{5,d}$ has the expected dimension.*

Theorem 3. *Set $d(5) := 4$. For all integers $n \geq 6$ define inductively the integer $d(n)$ in the following way. Let $d(n)$ be the first integer such that $d(n) \geq d(n-1) + 3$ and*

$$2(\binom{n+d(n)-3}{n} - \binom{n+d(n-1)+1}{n})/(2n-1) \geq \binom{n+d(n-1)+1}{n} + 4(d(n) - d(n-1)) . \quad (1)$$

Fix integers d, n, s such that $n \geq 6$, $d \geq d(n)$, $s > 0$. Let $Z \subset \mathbf{P}^n$ be a general union of s (2,3)-points. Then either $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ (case $(2n+1)s \leq \binom{n+d}{n}$) or $h^0(\mathbf{P}^4, \mathcal{I}_Z(d)) = 0$ (case $(2n+1)s \geq \binom{n+d}{n}$). Equivalently, for all (n, d, s) as above the $(s-1)$-th secant variety $S^{s-1}(X_{n,d})$ of $X_{4,d}$ has the expected dimension.

In the case $(n, d, s) = (4, 3, 4)$ the secant variety $S^3(X_{4,3})$ has not the expected dimension (see [6], Proposition 3.4, or [6], §4). Notice that Theorem 3 is a rather deep asympthotic result, in the spirit of [3]: for fixed n and large d all secant varieties of $X_{n,d}$ have the expected dimension. The reader will easily notice that if computer computations should give that the conjecture in [6] is true for $d = 3$ and, say, $n \leq 200$, a couple of pages would be sufficient to extend the non-asympotic part of this paper (i.e. Theorems 1 and 2) to the case $n \leq 200$. Even weaker results for $d = 3$ (and a fixed n) may be sufficient to prove that all secant varieties of $X_{n,d}$ for all $d \geq 4$ have the expected dimension. Indeed, in the case $n = 5$ (i.e. in the statement and proof of Theorem 2) we excluded the case $d = 3$, but from very weak informations for $d = 3$ we could prove the case $d = 4$ and then everything was easy. To check the conjecture for a single value of d, say $d = 13$, but all n seems to be harder.

2. Preliminary Results and Numerical Lemmas

For all positive integers n and d define the integers $s_{n,d}$ and $r_{n,d}$ by the relations

$$(2n+1)s_{n,d} + r_{n,d} = \binom{n+d}{n}, \quad 0 \leq r_{n,d} \leq 2n\,. \tag{2}$$

Set $s'_{n,d} = s_{n,d}$ if $r_{n,d} = 0$ and $s'_{n,d} = s_{n,d} + 1$ if $r_{n,d} > 0$. Notice that $s_{n,d} := \lfloor \binom{n+d}{n}/(2n+1) \rfloor$ and $s'_{n,d} = \lceil \binom{n+d}{n}/(2n+1) \rceil$. If a general union A of x (2,3)-points satisfies $h^1(\mathbf{P}^n, \mathcal{I}_A(d)) = 0$ (resp. $h^0(\mathbf{P}^n, \mathcal{I}_A(d)) = 0$), then $x \leq s_{n,d}$ (resp. $x \geq s'_{n,d}$).

Consider the following statements $A(n, d)$ and $B(n, d)$, $n \geq 2$, $d \geq 2$.

$A(n, d)$: Let $Z \subset \mathbf{P}^n$ be a general union of $s_{n,d}$ (2,3)-points of $\mathbf{P}^n$. Then $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$.

$B(n, d)$: Let $T \subset \mathbf{P}^n$ be a general union of $s'_{n,d}$ (2,3)-points of $\mathbf{P}^n$. Then $h^0(\mathbf{P}^n, \mathcal{I}_T(d)) = 0$.

Take Z (resp. W) as in $A(n, d)$ (resp. $B(n, d)$). Let A (resp. B) be any zero-dimensional scheme such that $A \subseteq Z$ (resp. $T \subseteq B$). If $A(n, d)$ is true, then $h^1(\mathbf{P}^n, \mathcal{I}_A(d)) = 0$. Hence $A(n, d)$ is true if and only if $X_{n,d}$ is s-ordinary for all $s \le s_{n,d}$. If $B(n,d)$ is true, then $h^0(\mathbf{P}^n, \mathcal{I}_B(d)) = 0$. Hence $B(n,d)$ is true if and only if $X_{n,d}$ is s-ordinary for all $s \ge s'_{n,d}$.

We will also use the following statements $A'(n,d)$ and $B'(n,d)$:

$A'(n,d)$: Fix a hyperplane $H \subset \mathbf{P}^n$. Let $Z \subset \mathbf{P}^n$ be a general union of $s_{n,d}$ (2,3)-points such that $2n-2$ of them have support contained in H and are associated to a line of H. Then $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$.

$B'(n,d)$: Fix a hyperplane $H \subset \mathbf{P}^n$. Let $T \subset \mathbf{P}^n$ be a general union of $s'_{n,d}$ (2,3)-points such that $2n-2$ of them have support contained in H and are associated to a line of H. Then $h^0(\mathbf{P}^n, \mathcal{I}_T(d)) = 0$.

The assumption on Z (resp. W) in $A'(n,d)$ (resp. $B'(n,d)$) means that $2n-2$ of the connected components of Z (resp. W) intersect H in a (2,3)-point of H. We will say that any such component has *strong support* on H. By semicontinuity $A'(n,d)$ implies $A(n,d)$ and $B'(n,d)$ implies $B(n,d)$.

Remark 1. Let $H \subset \mathbf{P}^n$ be a hyperplane and $P \in H$. Let Z be a $(2,3)$-point of $\mathbf{P}^n$ in the sense of [6], p. 977, supported by P. Hence length$(Z) = 2n+1$, $Z_{red} = \{P\}$ and there is a line $L \subset \mathbf{P}^n$ such that $P \in D$ and $\mathcal{I}_Z = (\mathcal{I}_{\{P\}})^3 + (\mathcal{I}_L)^2$. If $L \subset H$ (i.e. if Z is strongly supported by H), then $H \cap Z$ is a $(2,3)$-point of H and hence length$(Z \cap H) = 2n-1$ and length$(\mathrm{Res}_H(Z)) = 2$; more precisely, in this case $\mathrm{Res}_H(Z)$ is the first infinitesimal neighborhood of P in L. If $L \nsubseteq H$, then length$(Z \cap H) = n$, $Z \cap H$ is the first infinitesimal neighborhood of P in H, length$(\mathrm{Res}_H(Z)) = n+1$ and $\mathrm{Res}_H(Z)$ is the first infinitesimal neighborhood of P in $\mathbf{P}^n$; hence in this case $\mathrm{Res}_H(Z) \cap H$ is the first infinitesimal neighborhood of P in H and $\mathrm{Res}_H(\mathrm{Res}_H(Z)) = \{P\}$ (with its reduced structure). By semicontinuity to check that $X_{n,d}$ is s-ordinary for all s it is sufficient to find zero-dimensional schemes Z', W' of $\mathbf{P}^n$ such that $h^0(\mathbf{P}^n, \mathcal{I}_{Z'}(d)) = r_{n,d}$, $h^0(\mathbf{P}^n, \mathcal{I}_{W'}(d)) = 0$, Z' is a flat degeneration of a flat family of disjoint unions of $s_{n,d}$ $(2,3)$-points of $\mathbf{P}^n$ and W' is a flat degeneration of a flat family of unions of $s_{n,d}+1$ $(2,3)$-points of $\mathbf{P}^n$. Even more by the differential lemma proved in [3] (see [3], Proposition 8.2, Proposition 9.1 and Figure 1 at p. 308) as a virtual scheme Q (not an actual scheme) whose intersection $Q \cap H$ with H has length one, whose residual intersection $\mathrm{Res}_H(Q)$ has length $2n$, $\mathrm{Res}_H(Q) \cap H$ has length n and $\mathrm{Res}_H(\mathrm{Res}_H(Q))$ has lenght n, i.e. instead of the integers $(n+1, n)$ or $(n, n, 1)$ allowable by true schemes, we may use the integers $(1, n, n)$ (see Figure 1 at p. 308 of [3]). We will call it a specialization of type $(1, n, n)$ with respect to H and supported by the point $Q_{red} \in H$.

Remark 2. Fix an integer $d > 0$, a zero-dimensional scheme $Z \subset \mathbf{P}^n$ and a hyperplane $H \subset \mathbf{P}^n$. Assume that Z contains u reduced connected components and v length 2 connected component, such that their union E is a general union of u points of H and v tangent vectors of H. We stress that E must be general after fixing $F := Z\backslash E$. Set $\gamma := \binom{n+d}{n} - \mathrm{length}(Z)$. Fix non-negative integers α, β such that $\alpha \geq -\gamma$ and $\beta \geq \gamma$. We claim that to check that $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) \leq \alpha$ (resp. $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) \leq \beta$) it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_F(d)) \leq \alpha$ and $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(F)}}(d-1)) \leq \alpha + \gamma$ (resp. $h^0(\mathbf{P}^n, \mathcal{I}_F(d) \leq \beta + u + 2v$ and $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(F)}}(d-1)) \leq \beta$). Let U be the image of the restriction map $\rho : H^0(\mathbf{P}^n, \mathcal{I}_F(d)) \to H^0(H, \mathcal{I}_{F\cap H}(d))$. The generality of E (for fixed F) implies that E imposes $\min\{\dim(U), u + 2v\}$ independent conditions to U (see [7]). Hence to check the h^1-part it is sufficient to show that if $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(F)}}(d-1)) \leq \alpha + \gamma$, then $\dim(U) \geq u + 2v$. The assumption $h^1(\mathbf{P}^n, \mathcal{I}_F(d)) \leq \alpha$ is equivalent to $h^0(\mathbf{P}^n, \mathcal{I}_F(d) \leq \gamma + \alpha + u + 2v$, because $\mathrm{length}(F) = \mathrm{length}(Z) - u - 2v$. Notice that $\mathrm{Ker}(\rho) \equiv H^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(F)}}(d-1))$. Since $\alpha + \gamma \geq h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(F)}}(d-1))$, the proof of the h^1-part is over. Now we will check the h^0-part. Again, $\mathrm{Ker}(\rho) \equiv H^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(F)}}(d-1))$. Hence $\dim(\mathrm{Im}(\rho) = h^0(\mathbf{P}^n, \mathcal{I}_F(d)) - h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(F)}}(d-1))$. If $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(F)}}(d-1)) \leq h^0(\mathbf{P}^n, \mathcal{I}_F(d)) - u - 2v$, we are done. If $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(F)}}(d-1)) \geq h^0(\mathbf{P}^n, \mathcal{I}_F(d)) - u - 2v$, then the restriction map $\eta : H^0(\mathbf{P}^n, \mathcal{I}_Z(d)) \to H^0(H, \mathcal{I}_{Z\cap H}(d))$ is zero. Hence $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z)}}(d-1))$. Since $\mathrm{Res}_H(Z) = \mathrm{Res}_H(F)$, we are done. Alternatively, the h^0-part for an arbitrary $\beta \geq 0$, may be reduced to the case $\beta = 0$ taking the union of Z with β general points instead of Z. We do not need to check in advance the condition $\gamma \geq \beta$, because $\mathrm{length}(F) = \mathrm{length}(Z) - u - 2v = \binom{n+d}{d} - \gamma - u - 2v$ and hence $h^0(\mathbf{P}^n, \mathcal{I}_F(d) \leq \beta + u + 2v$ implies $\beta \geq \gamma$.

Lemma 1. *$s_{4,d} - s_{3,d} \leq s_{4,d-1}$ for all $d \geq 4$.*

Proof. Since $9(s_{4,d} - s_{4,d-1}) = \binom{d+4}{4} - \binom{d+3}{4} + r_{4,d-1} - r_{4,d} \leq \binom{d+3}{3} - 8$, while $7s_{3,d} \geq \binom{d+3}{3} - 6$, we are done for $d \geq 6$, while for $d = 4, 5$ we may use $s_{4,5} = 14$, $s_{3,5} = 8$, $s_{4,4} = 7$, $s_{4,3} = 3$ and $s_{3,4} = 5$. □

In the same way we see the following result.

Lemma 2. *$s'_{4,d} - s_{3,d} - r_{3,d} \leq s_{4,d-1}$ for all $d \geq 4$.*

Lemma 3. *We have $s_{4,d} \geq s_{3,d} + r_{3,d}$ for all $d \geq 5$.*

Proof. By (2) for $n = 4$ and $n = 3$ it is sufficient to check if

$$(\binom{d+4}{4} - 8)/9 \geq (\binom{d+3}{3} - 6)/7 + 6\,, \tag{3}$$

i.e. if

$$7\cdot\binom{d+4}{4} \ge 9\cdot\binom{d+3}{3} + 56 - 54 + 6\cdot 9\cdot 7\,. \tag{4}$$

Since $\binom{10}{4} = 210$ and $\binom{9}{3} = 78$, we immediately see that (4) is true for all $d \ge 6$. For $d = 5$ we use $s_{4,5} = 14$, $s_{3,5} = 8$ and $r_{3,5} = 0$. □

Lemma 4. *$s_{4,d} \ge s_{3,d} + s_{4,d-2}$ for all $d \ge 5$.*

Proof. We have $s_{4,d} - s_{4,d-2} = ((\binom{d+4}{4} - r_{4,d} - \binom{d+2}{4} + r_{4,d-2})/9 \ge ((d+2)(d+1)(2d+3)/24 - 8)/9 \ge (d+3)(d+2)(d+1)/(24\cdot 7)$ for all $d \ge 6$. For $d = 5$ use that $s_{4,5} = 14$, $s_{3,5} = 8$ and $s_{4,3} = 3$. □

Lemma 5. *$s_{5,d} - s_{4,d} \le s_{5,d-1}$ for all $d \ge 4$.*

Proof. Since $11(s_{5,d} - s_{5,d-1}) = \binom{d+5}{5} - \binom{d+4}{5} + r_{5,d-1} - r_{5,d} \le \binom{d+4}{4} - 10$, while $9s_{4,d} \ge \binom{d+4}{4} - 8$, we are done for $d \ge 6$, while for $d = 4, 5$ we may use $s_{5,5} = 22$, $s_{5,4} = 11$, $s_{5,3} = 5$, $s_{4,5} = 14$ and $s_{4,4} = 7$. □

In the same way we see the following result.

Lemma 6. *$s'_{5,d} - s_{4,d} - r_{4,d} \le s_{4,d-1}$ for all $d \ge 4$.*

Lemma 7. *We have $s_{5,d} \ge s_{4,d} + r_{4,d}$ for all $d \ge 5$.*

Proof. By (2) for $n = 5$ and $n = 4$ it is sufficient to check if

$$(\binom{d+5}{5} - 10)/11 \ge (\binom{d+4}{4} - 8)/7 + 6\,. \tag{5}$$

For low d use the explicit values: $s_{5,6} = 42$, $s_{5,5} = 22$, $s_{5,4} = 11$, $s_{4,6} = 23$, $r_{4,6} = 3$, $s_{4,5} = 14$, $r_{4,5} = 0$. □

Lemma 8. *$s_{5,d} \ge s_{4,d} + s_{5,d-2}$ for all $d \ge 5$.*

Proof. We have $s_{5,d} - s_{5,d-2} = ((\binom{d+5}{5} - r_{5,d} - \binom{d+3}{5} + r_{5,d-2})/11 \ge ((d+2)(d+1)(2d+3)/24 - 8)/9 \ge (d+4)(d+3)(d+2)(d+1)/(120\cdot 9)$ for all $d \ge 6$. For $d = 5$ use that $s_{5,5} = 22$, $s_{4,5} = 14$ and $s_{5,3} = 5$. □

Remark 3. Fix integers n, x, y such that $n \ge 2$ and $x > y > 0$. Using (2) for $n' := n - 1$ and that $\binom{n+i}{n} - \binom{n+i-1}{n} = \binom{n+i-1}{n-1}$ we get:

$$(2n-1)(\sum_{i=y}^{x} s_{n-1,i}) = \binom{n+x}{n} - \binom{n+y}{n} - \sum_{i=y}^{x} r_{n-1,i}\,, \tag{6}$$

$$\sum_{i=y}^{x} r_{n-1,i} \le (2n-2)(x-y)\,. \tag{7}$$

From (6) and 7) we get

$$\sum_{y=y}^{x} 2s_{n-1,i} - \sum_{i=y}^{x} r_{n-1,i} \geq 2(\binom{n+x}{n} - \binom{n+y}{n})/(2n-1) - 2(x-y)\,. \quad (8)$$

If $y = d(n-1)+1$ and $x \geq d(n)-2$, then the definition of $d(n)$ and (8) give

$$\sum_{y=y}^{x} 2s_{n-1,i} - \sum_{i=y}^{x} r_{n-1,i} \geq \binom{n+d(n-1)+1}{n}\,. \quad (9)$$

Lemma 9. *We have $s_{n,d} \geq s_{n-1,d} + r_{n-1,d}$ for all $d \geq 2n$.*

Proof. By (2) for n and $n' := n-1$ it is sufficient to check if

$$(\binom{n+d}{n} - 2n)/(2n+1) \geq (\binom{n+d-1}{n-1} - 2n+2)/(2n-1) + 2n - 2 \quad (10)$$

Notice that (11) is satisfied

$$(2n-1)\cdot\binom{n+d}{n} \geq (2n+1)\cdot\binom{n+d-1}{n-1} + (2n+1)(2n-2)^2 \quad (11)$$

Since $\binom{n+d}{n}/\binom{n+d-1}{n-1} = (n+d)/n$, (11) is satisfied for all $d \geq 2n$. $\square$

Lemma 10. *Fix an integer $n \geq 6$. Then $s_{n,d} - s_{n-1,d} \leq s_{n,d-1}$ for all $d \geq 2n$.*

Proof. Since $(2n+1)(s_{n,d} - s_{n,d-1}) = \binom{n+d}{n} - \binom{n+d-1}{n} + r_{n,d-1} - r_{n,d} \leq \binom{n+d-1}{n-1} - 2n$, while $(2n-1)s_{n-1,d} \geq \binom{n+d-1}{n-1} - 2n + 2$, we are done. $\square$

In the same way we see the following result.

Lemma 11. *$s'_{n,d} - s_{n-1,d} - r_{n-1,d} \leq s_{n,d-1}$ for all $d \geq 2n$.*

Lemma 12. *We have $s_{n,d} < \sum_{j=d(n-1)}^{d} s_{n-1,j}$ for all integers $d \geq d(n)$.*

Proof. Use (6) and the definition (1) of the integer $d(n)$. $\square$

3. Proofs of Theorems 1, 2 and 3

In this section we will prove Theorems 1, 2 and 3.

Proof of Theorem 1. Fix a hyperplane $H \subset \mathbf{P}^4$ and general $P_i \in H$, $i \geq 1$. Let Z (resp. T) be a general union of $s_{4,d}$ (resp. $s'_{4,d}$) (2,3)-points, where $s'_{4,d} = s_{4,d}$ if $r_{n,d} = 0$ and $s'_{4,d} = s_{4,d}$ if $r_{n,d} = 0$. Hence to check $A(4,d)$ (resp.

$B(4,d)$) it is sufficient to prove $h^1(\mathbf{P}^4, \mathcal{I}_Z(d)) = 0$ (resp. $h^0(\mathbf{P}^4, \mathcal{I}_T(d)) = 0$). Let W be the union of Z and $r_{4,d}$ general points. We recall that $B(4,3)$ is false (see [6], Proposition 3.4).

(a) Here we will check $A(4,3)$. Recall that $s_{4,3} = 3$, $r_{4,3} = 8$, $s_{3,3} = 2$ and $r_{3,3} = 6$. We specialize W to a general union U of 2 (2,3)-points with strong support on H, 6 points of H, one (2,3)-point and 2 points. By [6], 3.2, $h^i(H, \mathcal{I}_{U\cap H}(3)) = 0$. $\mathrm{Res}_H(U)$ is a general union of 2 tangent vectors of H, one (2,3) point and 2 points. By Remark 2 it is sufficient to prove that $h^1(\mathbf{P}^3, \mathcal{I}_A(2)) = h^0(\mathbf{P}^3, \mathcal{I}_A(1)) = 0$, where A is a general union of one (2,3)-point and 2 points. Since $\mathrm{length}(A) = 13 \le \binom{6}{2}$, it is sufficient to use the case $(n,d,s) = (n,2,1)$ of [6], Proposition 1.1.

(b) Here we will check $A(4,4)$. Recall that $s_{4,4} = 7$, $r_{4,4} = 7$, $s_{3,4} = 5$ and $r_{3,4} = 0$. We specialize Z to a general union U of 5 (2,3)-points with strong support on H and 2 (2,3)-points. By [4] $h^i(H, \mathcal{I}_{U\cap H}(3)) = 0$. Let A be a general union of 2 (2,3)-points. $\mathrm{Res}_H(U)$ is a general union of A and 5 tangent vectors of H. Hence $h^1(\mathbf{P}^4, \mathcal{I}_{\mathrm{Res}_H(U)}(3)) = 0$ by $A(4,3)$, Remark 2 and the case $(n,j,s) = (4,2,2)$ of [6], 3.2, which gives $h^0(\mathbf{P}^3, \mathcal{I}_A(2)) \le 7 = r_{4,4}$.

(c) Here we will check $B(4,4)$. We specialize T to a general union U of 5 (2,3)-points with strong support on H and 3 (2,3)-points. By [4], $h^i(H, \mathcal{I}_{U\cap H}(3)) = 0$. Let D (resp. D') be a general union of 3 (resp. 4) (2,3)-points. $\mathrm{Res}_H(U)$ is a general union of D and 5 tangent vectors of H. By Remark 2 it is sufficient to check $h^0(\mathbf{P}^4, \mathcal{I}_D(3)) \le 10$ and $h^0(\mathbf{P}^4, \mathcal{I}_D(2)) = 0$. The last vanishing is true by the case $(n,s) = (4,3)$ of [6], Proposition 3.4. $h^0(\mathbf{P}^4, \mathcal{I}_D(3)) \le 10$, because $h^0(\mathbf{P}^4, \mathcal{I}_D(3)) = 1$ (see [6], Proposition 3.4).

(d) Here we will check $A'(4,5)$ and $B'(4,5)$ and hence $A(4,5)$ and $B(4,6)$. Since $s_{4,5} = 14$ and $r_{4,5} = 0$, $A(4,5) = B(4,5)$ and $A'(4,5) = B'(4,5)$. Since $s_{3,5} = 8$ and $r_{3,5} = 0$, we specialize Z to a general union of 8 (2,3)-points with strong support on H. Then we go on as in part (c)

(e) Here we assume $d \ge 6$ and that $A'(4,j)$ and $B'(4,j)$ are true for all $5 \le j \le d-1$. Here we prove $A'(4,d)$. We degenerate Z to a general union Z' of $s_{4,d} - s_{3,d} - r_{3,d}$ (2,3)-points, $s_{3,d}$ (2,3)-points with strong support on H and the points P_i, $1 \le i \le r_{3,d}$, seen as virtual schemes obtained applying Remark 1 at each P_i, $1 \le i \le r_{3,d}$ with respect to the sequence $(1,4,4)$. Here we use that $s_{4,d} \ge s_{3,d} + r_{3,d}$ for all $d \ge 5$ (Lemma 3). Notice that $s_{3,d} \ge 6$ and hence Z' has at least 6 connected components strongly supported by H. Since $H \cap Z'$ is a general union of $s_{3,d}$ (2,3)-points of H and $r_{3,d}$ general points of H, $h^i(H, \mathcal{I}_{Z'\cap H}(d)) = 0$, $i = 0,1$ (see [4], Theorem 2). $\mathrm{Res}_H(Z')$ is a general union of A and $s_{3,d}$ tangent vectors of H, where A is a general union of $s_{4,d} - s_{3,d} - r_{3,d}$ (2,3)-points and $r_{3,d}$ length 8 schemes supported by the points $P_1, \dots, P_{r_{3,d}}$ and with type $(4,4)$ with respect to H (Remark 1). By Remark 2 it is sufficient to prove $h^1(\mathbf{P}^4, \mathcal{I}_A(d-1)) = 0$ and $h^0(\mathbf{P}^4, \mathcal{I}_{\mathrm{Res}_H(A)}(d-2)) = 0$. We will first prove the h^1-part. Let $B \subset \mathbf{P}^4$ be a union of $s_{4,d} - s_{3,d}$ disjoint (2,3)-points such that $A \subseteq B$: each length 8 connected component of A is contained in at least one (2,3)-point of $\mathbf{P}^4$ with the same support. Since any 4 points of $\mathbf{P}^4$ are contained in a hyperplane, if $r_{3,d} \le 4$, then B may be considered as a general union of $s_{4,d} - s_{3,d}$ (2,3)-points of $\mathbf{P}^4$. Since $s_{4,d} - s_{3,d} \le s_{4,d-1}$ (Lemma 1) we may apply A(4,d-1) and get $h^1(\mathbf{P}^4, \mathcal{I}_A(d-1)) \le h^1(\mathbf{P}^4, \mathcal{I}_B(d-1)) = 0$. If $r_{3,d} \ge 5$, then we use $A'(4, d-1)$. Now we will check that $h^0(\mathbf{P}^4, \mathcal{I}_{\mathrm{Res}_H(A)}(d-2)) = 0$. $\mathrm{Res}_H(A)$ contains a disjoint union C of $s_{4,d} - s_{3,d}$ general (2,3)-points. Since $s_{4,d} - s_{3,d} \ge s_{4,d-2}$ (Lemma 4) if $d \ge 6$ we may apply $B(4, d-2)$.

(f) Here we assume $d \ge 6$ and that $A'(4,j)$ and $B'(4,j)$ are true for all $5 \le j \le d-1$. Here we prove $B'(4,d)$. By part (e) we may assume $s'_{4,d} = s_{4,d}+1$, i.e. $r_{4,d} > 0$. We degenerate T to a general union W' of $s'_{4,d} - s_{3,d} - r_{3,d}$ (2,3)-points, $s_{3,d}$ (2,3)-points with strong support on H and associated to a line contained in H and the points P_i, $1 \le i \le r_{3,d}$, seen as virtual schemes obtained applying Remark 1 at each P_i, $1 \le i \le r_{3,d}$ with respect to the sequence $(1,4,4)$. Here we use that $s_{4,d} \ge s_{3,d} + r_{3,d}$ for all $d \ge 5$ (Lemma 4). Since $H \cap W'$ is a general union of $s_{3,d}$ (2,3)-points of H and $r_{3,d}$ general points of H, $h^i(H, \mathcal{I}_{W' \cap H}(d)) = 0$, $i = 0, 1$ (see [4], Theorem 2). $\mathrm{Res}_H(W')$ is a general union of A and $s_{3,d}$ tangent vectors of H, where A is a general union of $s'_{4,d} - s_{3,d} - r_{3,d}$ (2,3)-points and $r_{3,d}$ length 8 schemes supported by the points $P_1, \dots, P_{r_{3,d}}$ and with type $(4,4)$ with respect to H (Remark 1). By Remark 2 it is sufficient to prove $h^0(\mathbf{P}^4, \mathcal{I}_A(d-1)) \le 2s_{3,d}$ and $h^0(\mathbf{P}^4, \mathcal{I}_{\mathrm{Res}_H(A)}(d-2)) = 0$. The proof of the second vanishing was done in part (e). Now we will check that $h^0(\mathbf{P}^4, \mathcal{I}_A(d-1)) \le 2s_{3,d}$, i.e that $h^1(\mathbf{P}^4, \mathcal{I}_A(d-1)) \le 9 - r_{4,d}$. We will check that $h^1(\mathbf{P}^4, \mathcal{I}_A(d-1)) = 0$. Let $B \subset \mathbf{P}^4$ be a union of $s'_{4,d} - s_{3,d}$ disjoint (2,3)-points such that $A \subseteq B$: each length 8 connected component of A is contained in at least one (2,3)-point of $\mathbf{P}^4$ with the same support. Since $s'_{4,d} - s_{3,d} \le s_{4,d-1}$ (Lemma 2) and $r_{3,d} \le 6$, $A'(4, d-1)$ gives $h^1(\mathbf{P}^4, \mathcal{I}_A(d-1)) = 0$, concluding the proof of Theorem 1. □

Proof of Theorem 2. Fix a hyperplane $H \subset \mathbf{P}^5$. Let Z (resp. T) be a general union of $s_{5,d}$ (resp. $s'_{5,d}$) (2,3)-points.

(a) Let $A \subset \mathbf{P}^5$ be a general union of 4 (2,3)-points. Here we will prove $h^1(\mathbf{P}^5, \mathcal{I}_A(3)) = 0$. We degenerate A to a general union A' of 3 (2,3)-points strongly support by H and one general (2,3)-point. $A(4,3)$ (checked in part (a) of the proof of Theorem 1) gives $h^1(H, \mathcal{I}_{A' \cap H}(3)) = 0$. $\mathrm{Res}_H(A')$ is a general union of one (2,3)-point (call it B) and 3 general tangent vectors of H. By

Remark 2 it is sufficient to prove $h^1(\mathbf{P}^5, \mathcal{I}_B(2)) = 0$ and $h^0((\mathbf{P}^5, \mathcal{I}_B(1)) = 0$. The first vanishing is true, because $X_{5,2}$ has the expected dimension (see [6], Proposition 1.1). The second vanishing is obvious, because any (2,3)-point spans the ambient projective space.

(b) Let $A \subset \mathbf{P}^5$ be a general union of 8 (2,3)-points. Here we will prove $h^0(\mathbf{P}^5, \mathcal{I}_A(3)) = 0$. We degenerate A to a general union A' of 3 (2,3)-points and 5 (3,2)-points strongly supported by H. Remember that $B(4,3)$ is false, but $s'_{4,3} = 4$ and the proof of $A(4,3)$ easily gives $H^0(H, \mathcal{I}_{H\cap A'}(3)) = 0$. $\mathrm{Res}_H(A')$ is a general union of 3 (2,3)-points and 5 general tangent vectors of H. Let B denote the union of the length 11 components of $\mathrm{Res}_H(A')$. Since $B \cap H = \emptyset$, Remark 2 shows that it is sufficient to prove $h^0(\mathbf{P}^5, \mathcal{I}_B(2)) \leq 10$ and $h^0((\mathbf{P}^5, \mathcal{I}_B(1)) = 0$. $h^0(\mathbf{P}^5, \mathcal{I}_B(2)) = 0$ by part i) of [6], Proposition 3.3. The second vanishing is obvious, because any (2,3)-point spans the ambient projective space.

(c) Here we will prove $A(5,4)$. We specialize Z to a general union Z' of 7 (2,3)-points strongly supported by H, two virtual schemes obtained applying Remark 1 at (P_1, H) and (P_2, H) and 2 general (2,3)-points. $A(4,4)$ gives $h^1(H, \mathcal{I}_{Z'\cap H}(4)) = 0$. Let B denote the set of all unreduced components of $\mathrm{Res}_H(Z')$. Since $\mathrm{Res}_H(Z')\backslash B$ is a general union of 7 tangent vectors of H, it is sufficient to prove $h^1(\mathbf{P}^5, \mathcal{I}_B(3)) = 0$ and $h^0(\mathbf{P}^5, \mathcal{I}_{\mathrm{Res}_H(B)}(2)) = 0$. Let A denote the only union of 4 (2,3)-points containing B. Since any two lines of $\mathbf{P}^5$ are contained in a hyperplane, A may be seen as a general union of 4 (2,3)-points of H. Hence $h^1(\mathbf{P}^5, \mathcal{I}_B(3)) \leq h^1(\mathbf{P}^5, \mathcal{I}_A(3)) = 0$ by part (a). $\mathrm{Res}_H(B)$ is a general union of 2 (2,3)-points and two double points of H (not double points with support on H). Let E denote the union of the two length 11 components of $\mathrm{Res}_H(B)$. $h^0(\mathbf{P}^5, \mathcal{I}_E(2)) = 3$ (see [6], Proposition 3.2). Since we may change H and the two length 5 connected components of $\mathrm{Res}_H(B)$ after fixing E, we immediately get $h^0(\mathbf{P}^5, \mathcal{I}_{\mathrm{Res}_H(B)}(2)) = 0$ (alternatively: use tangent vectors instead of length 5 components and then apply [7]).

(d) Here we will prove $B(5,4)$. We specialize T to a general union W' of 4 (2,3)-points and 8 (2,3)-points strongly supported by H. $B(4,4)$ gives $h^0(H, \mathcal{I}_{W'\cap H}(4)) = 0$. Let A be the union of the length 11 components of $\mathrm{Res}_H(W')$. Hence $A \cap H = \emptyset$ and A is a general union of 4 (2,3)-points. Since $\mathrm{Res}_H(W')\backslash A$ is a general union of 8 tangent vectors of H and $A \cap H = \emptyset$, Remark 2 shows that it is sufficient to prove $h^0(\mathbf{P}^5, \mathcal{I}_A(3)) \leq 16$ and $h^0(\mathbf{P}^5, \mathcal{I}_A(2) = 0$. Part (a) gives $h^0(\mathbf{P}^5, \mathcal{I}_A(3)) = 14$. Since $2\cdot 4 > 5 = n$, part ii) of [6], Proposition 3.3, gives $h^0(\mathbf{P}^5, \mathcal{I}_A(2) = 0$.

(e) Here we will prove $A'(5,5)$. We specialize Z to a general union Z' of 8 (2,3)-points and 14 (2,3)-points strongly supported by H. $A(4,5)$ gives $h^i(H, \mathcal{I}_{Z'\cap H}(5)) = 0$ for $i = 0, 1$. Notice that $14 \geq 8$ and hence Z' may be used to prove $A'(5,5)$, not just $A(5,5)$. $\mathrm{Res}_H(Z')$ is a general union of 14 tangent vectors of H and a general union A of 8 (2,3)-points. Since $A \cap H = \emptyset$ and $r_{5,5} = 10$, Remark 2 shows that it is sufficient to prove $h^1(\mathbf{P}^5, \mathcal{I}_A(4)) = 0$ and $h^0(\mathbf{P}^5, \mathcal{I}_A(3) \le 10$. $h^1(\mathbf{P}^5, \mathcal{I}_A(4)) = 0$, becuse $8 \le 11 = s_{5,4}$ and we proved $A(5,4)$ in part (c). $h^0(\mathbf{P}^5, \mathcal{I}_A(3)) = 0$ by part (b).

(f) Here we will prove $B'(5,5)$. We specialize T to a general union W' of 9 (2,3)-points and 14 (2,3)-points strongly supported by H. $A(4,5)$ gives $h^i(H, \mathcal{I}_{Z'\cap H}(5)) = 0$ for $i = 0,1$. $\mathrm{Res}_H(Z')$ is a general union of 14 tangent vectors of H and a general union A of 9 (2,3)-points. We continue as in part (e).

(g) Here we will prove $A'(5,6)$. Since $r_{5,6} = 0$, $B'(5,6) = A'(5,6)$. Since $s_{4,6} = 23$ and $r_{4,6} = 3$, we specialize Z to a general union Z' of 16 (2,3)-points, 23 (2,3)-points strongly supported by H and 3 virtual schemes obtained applying Remark 1 at 3 general points of H with respect to the sequence $(1,5,5)$. $A(4,6)$ gives $h^i(H, \mathcal{I}_{Z'\cap H}(6)) = 0$ for $i = 0,1$. $\mathrm{Res}_H(Z')$ is a general union of the general unionof 16 (2,3)-points, 3 length 10 schemes and 23 general tangent vectors of H. Let E be the union of the connected components of $\mathrm{Res}_H(Z')$ with length ≥ 10. It is sufficient to prove $h^1(\mathbf{P}^5, \mathcal{I}_E(5)) = 0$ and $h^0(\mathbf{P}^5, \mathcal{I}_{\mathrm{Res}_H(E)}(4)) = 0$. E is contained in a general union F of 19 (2,3)-points such that exactly 3 of them have support on H. Since $19 \le 22 = s_{5,5}$, $A'(5,5)$ gives $h^1(\mathbf{P}^5, \mathcal{I}_E(5)) \le h^1(\mathbf{P}^5, \mathcal{I}_F(5)) = 0$. $\mathrm{Res}_H(E)$ is a general union of 3 double points of H and a general union G of 16 (2,3)-points. $h^1(\mathbf{P}^5, \mathcal{I}_G(5)) = 0$, because $s'_{5,4} = 12$ and $B(5,4)$ is true (part (d)).

(h) Here we assume $d \ge 7$ and that $A'(5,t)$ and $B'(5,t)$ are true for all $5 \le t \le d-1$. Here we will prove $A'(5,d)$ and $B'(5,d)$. We copy word-for-word the proofs of parts (e) and (f) of Theorem 1, just changing the quotations to the numerical lemmas, i.e. quoting Lemma 8 (resp. 7, resp. 5, resp. 6) instead of Lemma 4 (resp. 3, resp. 1, resp. 2). □

Proof of Theorem 3. We assume $n \ge 6$ and use induction on n. We assume $A(n-1,t)$ and $B(n-1,t)$ for all $t \ge d(n-1)$. This assumption is satisfied for $n = 6$ by Theorem 2 and our definition of the integer $d(n-1)$. We will prove $A'(n,d)$ and $B'(n,d)$ for all $n \ge 6$ and $d \ge d(n)$. This is obviously enough to prove Theorem 3. We fix a hyperplane H of $\mathbf{P}^n$ and general $P_i \in H$, $i \ge 1$. We fix n,d such that $n \ge 6$ and $d \ge d(n)$. By the proof of Theorem 2 and induction on n we may assume that Theorem 3 is true for the integer $n' := n-1$. Let $Z \subset \mathbf{P}^n$ (resp. $T \subset \mathbf{P}^n$) be a general union of $s_{n,d}$ (resp. $s'_{n,d}$) (2,3)-points. It is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ and $h^0(\mathbf{P}^n, \mathcal{I}_T(d)) = 0$.

(a) We degenerate Z to a general union Z' of $s_{n,d} - s_{n-1,d} - r_{n-1,d}$ (2,3)-points, $s_{n-1,d}$ (2,3)-points strongly supported by H and $r_{n-1,d}$ virtual schemes obtained applying Remark 1 at (P_i, H), $1 \le i \le r_{n-1,d}$, with respect to the sequence $(1, n, n)$. Here we use that $s_{n,d} - s_{n-1,d} - r_{n-1,d} \ge 0$ (Lemma 9). Notice that $s_{n-1,d} \ge 2n-2$ and hence if $h^1(\mathbf{P}^n, \mathcal{I}_{Z'}(d)) = 0$, then $A'(n,d)$ is true. By $A(n-1,H)$ we have $h^i(H, \mathcal{I}_{Z' \cap H}(d)) = 0$, $i = 0, 1$. Let A be the union of all the connected components of length ≥ 3 of $\mathrm{Res}_H(Z')$. Since $\mathrm{Res}_H(Z') \backslash A$ is a general union of $r_{n-1,d}$ tangent vector of H, Remark 2 gives that to prove $h^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(Z')}(d-1)) = 0$ and hence $h^1(\mathbf{P}^n, \mathcal{I}_{Z'}(d)) = 0$ it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_A(d-1)) = 0$ and $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(A)}(d-2)) \le r_{n,d}$. A is contained in a unique union B of $s_{n,d} - s_{n-1,d}$ (2,3)-points, exactly $r_{n-1,d}$ of them supported by H. To check that $h^1(\mathbf{P}^n, \mathcal{I}_A(d-1)) = 0$ it is sufficient to check that $h^1(\mathbf{P}^n, \mathcal{I}_B(d-1)) = 0$. First assume $s_{n,d} - s_{n-1,d} \ge s_{n-1,d-1} + r_{n-1,d-1}$; if this condition is not satisfied we will say that we stopped at $h^1(\mathcal{I}(d-1))$. We degenerate B to a general union B' of $s_{n,d} - s_{n-1,d} - s_{n-1,d-1} - r_{n-1,d-1}$ (2,3)-points and $s_{n-1,d-1}$ (2,3)-points strongly supported by H; among the strongly supported components we take all the $r_{n-1,d}$ connected components of B intersecting H. $A(n-1, d-1)$ gives $h^i(H, \mathcal{I}_{B' \cap H}(d)) = 0$, $i = 0, 1$. Let A_1 be the union of the connected components of length ≥ 3 of $\mathrm{Res}_H(B')$. Since $\mathrm{length}(\mathrm{Res}_H(B')) = (2n+1)(s_{n,d} - s_{n-1,d}) - \binom{n+d-2}{n-1}$, to prove $h^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(B')}(d-2)) = 0$ and hence to prove $h^1(\mathbf{P}^n, \mathcal{I}_B(d-1)) = 0$ it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_{A_1}(d-2)) = 0$ and $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(B')}(d-3)) \le \binom{n+d-1}{n} - (2n+1)(s_{n,d} - s_{n-1,d})$. Then we continue, each time we find a new h^1-vanishing and a new h^0-inequality, unless, perhaps, we stop at a degree k because the union B_{d-k+1} of (2,3)-points obtained at that level has at most $s_{n-1,k} + r_{n-1,k} - 1$ connected components. If the latter case occurs, then we will say that we stopped at $h^1(\mathcal{I}(k))$.

(b) Take the stopping time k and $D := B_{d-k+1}$ as above. D is a union of z (2,3)-points, at most $r_{n-1,k+1} \le 2n$ of them intersecting H. By assumption $z < s_{n-1,k} + r_{n-1,k}$. First assume $s_{n-1,k} \le z < s_{n-1,k} + r_{n-1,k}$. We degenerate D to a general union of $s_{n-1,k}$ (2,3)-points strongly supported by H and $z - s_{n-1,k}$ virtual schemes obtained applying Remark 1 at general points of H with respect to the sequence $(1, n, n)$. Since $k \ge d(n-1)$, we have $h^1(H, \mathcal{I}_{D' \cap H}(k)) = 0$. Hence it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(D')}(k-1)) = 0$. Since $\mathrm{Res}_H(D')$ is a general union of certain $z - s_{n-1,k}$ length $2n$ schemes and $s_{n-1,k}$ general tangent vectors of H we conclude using Remark 2. Now assume $z \le s_{n-1,k}$. In this case we degenerate D to a general union D'' of z (2,3)-points strongly supported by H and immediately conclude by $A(n-1,k)$ and Remark 2.

(c) Now assume that we never stopped, at least until the integer $k := d(n-1) - 1$ for which we cannot use $A(n-1,k)$. At the degree $k := d(n-1) - 1$

we needed to check an h^1-vanishing for a zero-dimensional scheme D with $s_{n,d} - \sum_{j=k+1}^{d} s_{n-1,j}$ connected components. Since $s_{n,d} < \sum_{j=d(n-1)}^{d} s_{n-1,j}$ (Lemma 12), we got a contradiction.

(d) At each step in part (a) we got a new h^0-inequality to check. Take an integer t such that $n(d-1) \le t \le d-2$. We got an inequality $h^0(\mathbf{P}^n, \mathcal{I}_E(t)) \le \alpha$, where E is the union of the length ≥ 3 components of a scheme $\mathrm{Res}_H(F)$. E is a general union of, a (2,3)-points and b length $2n$ schemes supported by a point of H. We have $b \le r_{n-1,t+1}$ and in particular $b \le 2n-2$. We distinguish between the case $t = d-2$ and the case $t \le d-3$. First assume $t \le d-3$. In this case $\alpha \ge 2n-2$. There is a unique union F of $a+b$ (2,3)-points such that $E \subseteq F$. To check $h^0(\mathbf{P}^n, \mathcal{I}_E(t)) \le \alpha$ it is sufficient to check $h^0(\mathbf{P}^n, \mathcal{I}_F(t)) \le \alpha - b$. We first assume $a+b \ge s_{n-1,t} + r_{n-1,t}$. We degenerate F to a general union F' of $a+b-s_{n-1,t}-r_{n-1,t}$ (2,3)-points, $s_{n-1,t}$ (2,3)-points with support on H and $r_{n-1,t}$ virtual schemes obtained applying Remark 1 at general points of H with respect to the sequence $(1,n,n)$. Let F_1 be the union of the components of length ≥ 3 of $\mathrm{Res}_H(F')$. We reduce to prove $h^0(\mathbf{P}^n, \mathcal{I}_{F_1}(t-1)) \le \alpha - b + 2s_{n-1,t}$ and $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(F_1)}(t-2)) \le \alpha - b$. Let F_2 be the only union of $a+b-s_{n-1,t}$ (2,3)-points containing F_1. To check that $h^0(\mathbf{P}^n, \mathcal{I}_{F_1}(t-1)) \le \alpha - b + 2s_{n-1,t}$ it is sufficient to check $h^0(\mathbf{P}^n, \mathcal{I}_{F_2}(t-1)) \le \alpha - b + 2s_{n-1,t} - r_{n-1,t}$. Notice that $\alpha - b - (\binom{n+t}{n} - \mathrm{length}(F)) = (\alpha - b + 2s_{n-1,t} - r_{n-1,t} - (\binom{n+t-1}{n} - \mathrm{length}(F_2)$. We also got inductively the integer $\alpha - b$ in a similar way from the step $t+1 \mapsto t$ and hence at the end we reduce the problem to an inequality $\sum_{j=d(n-1)+1}^{d-3} s_{n-1,j} \ge (2n-2)(d-2-d(n-1)) + \binom{n+d(n-1)}{n-1}$ as in the inequality (9). More precisely, we must stop arriving at an empty scheme at a certain integer $k \ge d(n-1)+1$ and for that integer k the inequality to be checked is of the form $h^0(\mathbf{P}^n, \mathcal{I}_\Gamma(k)) \le \beta$ with $\beta \ge \binom{n+k}{n}$. Now we check the missing case $t = d-2$. Call E' (resp. E'') the union of the length $2n+1$ (resp. $2n$) connected components of E. We fix a general hyperplane M of $\mathbf{P}^n$ and we specialize E to a general union G of E'', $a - s_{n-1,d-2} - r_{n-1,d-2}$ (2,3)-points, $s_{n-1,d-2}$ (2,3)-points strongly supported by M and $r_{n-1,d-2}$ virtual schemes obtained applying Remark 1 with respect to M at general points of M. Let G_1 be the union of the components of $\mathrm{Res}_M(G)$ with length ≥ 3. Now we are in a situation with $\alpha \ge 4n-4 \ge b + r_{n-1,d}$. Let G_2 be the union of $a+b-s_{n-1,d-2}$ (2,3)-points containing G_1. Now we use G_2 instead of F for the integer $t := d-3$, using any of the two hyperplanes H or M.

(e) Here we will show how to modify the proof just given for Z to obtain $h^0(\mathbf{P}^n, \mathcal{I}_T(d)) = 0$. We degenerate T to a general union T' of $s'_{n,d} - s_{n-1,d} - r_{n-1,d}$ (2,3)-points, $s_{n-1,d}$ (2,3)-points strongly supported by H and $r_{n-1,d}$ virtual schemes obtained applying Remark 1 at (P_i, H), $1 \le i \le r_{n-1,d}$, with respect to the sequence $(1, n, n)$. Here we use that $s'_{n,d} - s_{n-1,d} - r_{n-1,d} \ge 0$ (Lemma 9). By $A(n-1, H)$ we have $h^i(H, \mathcal{I}_{T' \cap H}(d)) = 0$, $i = 0, 1$. Let A be the union of all the connected components of length ≥ 3 of $\mathrm{Res}_H(T')$. Since $\mathrm{Res}_H(Z') \backslash A$ is a general union of $r_{n-1,d}$ tangent vectors of H, Remark 2 gives that to prove $h^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z')}}(d-1)) = 0$ and hence $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) \le 2s_{n-1,d}$ it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_A(d-1)) = 0$ and $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(A)}}(d-2)) = 0$. From now on we continue as in parts (a), (b) and (c). We only need the h^0-inequalities considered in part (d). □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] B. Ådlandsvik, Joins and higher secant varieties, *Math. Scand.*, **61** (1987), 213-222.

[2] J. Alexander, A. Hirschowitz, Polynomial interpolation in several variables, *J. of Alg. Geom.*, **4** (1995), 201-222.

[3] J. Alexander, A. Hirschowitz, An asymptotic vanishing theorem for generic unions of multiple points, *Invent. Math.*, **140**, No. 2 (2000), 303-325.

[4] E. Ballico, On the secant varieties to the tangent developable of a Veronese variety, *J. Algebra*, **288**, No. 2 (2005), 279-286.

[5] A. Bernardi, M. V. Catalisano, A. Gimigliano, M. Idà, Osculating varieties of Veronese varieties and their higher secant varieties, *Canadian J. Math.*, **59**, No. 3 (2007), 488-502.

[6] M.V. Catalisano, A.V. Geramita, A. Gimigliano, On the secant variety to the tangential varieties of a Veronesean, *Proc. Amer. Math. Soc.*, **130**, No. 4 (2001), 975-985.

[7] C. Ciliberto, R. Miranda, Interpolations on curvilinear schemes, *J. Algebra*, **203**, No. 2 (1998), 677-678.

[8] A. Hirschowitz, La méthode d'Horace pour l'interpolation à plusieurs variables, *Manuscripta Math.*, **50** (1985), 337-378.