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ON THE $L-(p,q)$-TH ORDER OF WRONSKIANS

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Abstract: In the paper we study the relationship between the $L-(p,q)$-th order of a transcendental meromorphic function and that of a wronskian generated by it, where p,q are positive integers and $p>q$.

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Key Words: Wronskian, transcendental meromorphic function, slowly changing function, $L-$order, $L-$lower order, (p,q)-th order, (p,q)-th lower order, $L-(p,q)$-th order, $L-(p,q)$-th lower order

1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [5] and [1]. In the sequel we use the following notation: $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1,2,3,....$ and $\log^{[0]} x = x$.

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The following definitions are well known.

Definition 1. A meromorphic function $a = a(z)$ is called small with respect to f if $T(r,a) = S(r,f)$.

Definition 2. Let $a_1, a_2, ...a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ...a_k, f)$ the Wronskian determinant of $a_1, a_2, ..., a_k, f$ i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1^{'} & a_2^{'} & . & . & . & a_k^{'} & f^{'} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Definition 3. If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

is called the Nevanlinna deficiency of the value a.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a;f) > 0$ is countable and $\sum\limits_{a\neq\infty} \delta(a;f) + \delta(\infty;f) \leq 2$ (cf.[$1, p.43$]). If in particular, $\sum\limits_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, we say that f has the maximum deficiency sum.

Definition 4. The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Somasundaram and Thamizharasi [4] introduced the notions of L-order and L-lower order for entire functions, where $L = L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant $'a'$. Their definitions are as follows:

Definition 5. (see [4]) The L-order ρ_f^L and the L-lower order λ_f^L of an entire function f are defined as follows:

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log [rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log [rL(r)]}.$$

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log [rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log [rL(r)]}.$$

Juneja, Kapoor and Bajpai [2] defined the (p,q)-th order and (p,q)-th lower order of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} \quad \text{and} \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r},$$

where p,q are positive integers and $p>q$.

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \quad \text{and} \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where p,q are positive integers and $p>q$.

So with the help of the above notion one can easily define the L-(p,q)-th order and L-(p,q)-th lower order of entire and meromorphic functions.

Definition 6. The L-(p,q)-th order $\rho_f^L(p,q)$ and the L-(p,q)-th lower order $\lambda_f^L(p,q)$ of an entire function f are defined as

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} [rL(r)]} \quad \text{and} \quad \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} [rL(r)]},$$

where p,q are positive integers and $p>q$.

When f is meromorphic, one can easily verify that

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} [rL(r)]} \quad \text{and} \quad \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} [rL(r)]},$$

where p,q are positive integers and $p>q$.

The more generalised concept of L-(p,q)-th order and L-(p,q)-th lower order of entire and meromorphic functions are L^*-(p,q)-th order and L^*-(p,q) th lower order respectively. In order to prove our results we require the following definitions:

Definition 7. The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log \left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log \left[re^{L(r)}\right]}.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}.$$

Definition 8. The L^*-(p,q)-th order $\rho_f^{L^*}(p,q)$ and the L^*-(p,q)-th lower order $\lambda_f^{L^*}(p,q)$ of an entire function f are defined as

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}\left[re^{L(r)}\right]},$$

where p,q are positive integers and $p>q$.
When f is meromorphic, one can easily verify that

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]},$$

where p,q are positive integers and $p>q$.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper we establish the relationship between the L-(p,q)-th order of a transcendental meromorphic function and that of a Wronskian generated by it.

2. Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 1. (see [3]) *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$\lim_{r\to\infty} \frac{T(r,L(f))}{T(r,f)} = 1+k-k\delta(\infty;f).$$

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then the L-(p,q)-th order of $L(f)$ and that of f are same, where p,q are positive integers and $p>q$.*

Proof. By Lemma 1, $\lim\limits_{r\to\infty}\frac{\log^{[p]}T(r,L(f))}{\log^{[p]}T(r,f)}$ exists and is equal to 1.

$$\begin{aligned}
\text{So } \rho_{L(f)}^{L}(p,q) &= \limsup_{r\to\infty}\frac{\log^{[p]}T(r,L(f))}{\log^{[q]}[rL(r)]}\\
&= \limsup_{r\to\infty}\left\{\frac{\log^{[p]}T(r,f)}{\log^{[q]}[rL(r)]}.\frac{\log^{[p]}T(r,L(f))}{\log^{[p]}T(r,f)}\right\}\\
&= \limsup_{r\to\infty}\frac{\log^{[p]}T(r,f)}{\log^{[q]}[rL(r)]}.\lim_{r\to\infty}\frac{\log^{[p]}T(r,L(f))}{\log^{[p]}T(r,f)}\\
&= \rho_f^{L}(p,q).1=\rho_f^{L}(p,q).
\end{aligned}$$

This proves the theorem. □

Theorem 2. *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then the L-(p,q)-th lower order of $L(f)$ and that of f are equal, where p,q are positive integers and $p>q$.*

We omit the proof of Theorem 2 because it can be carried out in the line of Theorem 1.

In the following theorem we establish the relationship between the L^*-(p,q)-th order of $L(f)$ and that of f.

Theorem 3. *If f be a transcendental meromorphic function having the maximum deficiency sum. Then the L^*-(p,q)-th order of $L(f)$ and that of f are same, where p,q are positive integers and $p>q$.*

Proof. In view of Lemma 1, $\lim\limits_{r\to\infty}\frac{\log^{[p]}T(r,L(f))}{\log^{[p]}T(r,f)}$ exists and is equal to 1.

$$\begin{aligned}
\text{So } \rho_{L(f)}^{L^*}(p,q) &= \limsup_{r\to\infty}\frac{\log^{[p]}T(r,L(f))}{\log^{[q]}\left[re^{L(r)}\right]}\\
&= \limsup_{r\to\infty}\left\{\frac{\log^{[p]}T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]}.\frac{\log^{[p]}T(r,L(f))}{\log^{[p]}T(r,f)}\right\}\\
&= \limsup_{r\to\infty}\frac{\log^{[p]}T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]}.\lim_{r\to\infty}\frac{\log^{[p]}T(r,L(f))}{\log^{[p]}T(r,f)}\\
&= \rho_f^{L^*}(p,q).1=\rho_f^{L^*}(p,q).
\end{aligned}$$

Thus the theorem is established. □

Theorem 4. *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then the L^*-(p,q)-th lower order of $L(f)$ and that of f are equal, where p,q are positive integers and $p>q$.*

We omit the proof of Theorem 4 because it can be carried out in the line of Theorem 3.

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