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# WELL-POSEDNESS OF $M/G/1$ QUEUEING MODEL WITH SECOND OPTIONAL SERVICE

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**Abstract:** By using the Hille-Yosida Theorem, Phillips Theorem and Fattorini Theorem in functional analysis, the existence and uniqueness of positive time-dependent solution of $M/G/1$ queueing model with second optional service is obtained.

**AMS Subject Classification:** 47D03, 47N20
**Key Words:** $M/G/1$ queue with second optional service, time-dependent solution, $C_0-$semigroup, conservative operator

## 1. Introduction

In real life, we see such queue: one encounters numerous examples of the queuing situations where all arriving customers require the main service and only some may require the subsidiary service provided by the server. This queue is called $M/G/1$ queue with second optional service. In 2000, K.C. Madan first considered the queue and established the corresponding queueing model by supplementary variable technique (see [4]). He studied the time-dependent solution of the model by using Laplace transform and got expression of the probability generating function of the model. Roughly speaking, he obtained existence of the time-dependent solution of the model. Moreover, he studied steady-state solution and other indices of the queue in the steady-state. In this paper, we first introduce suitable state space and operator corresponding to the model, next convert the model into an abstract Cauchy problem in the state space, last

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by using the Hille-Yosida Theorem, Phillips Theorem and Fattorini Theorem prove that the $M/G/1$ queuing model with second optional service has a unique positive time-dependent solution which satisfies probability condition.

The $M/G/1$ queuing model with second optional service can be written as (see [4]):

$$\frac{dQ(t)}{dt} = -\lambda Q(t) + \mu_2 P_0^{(2)}(t) + (1-r)\int_0^\infty \mu_1(x)P_0^{(1)}(x,t)dx, \tag{1}$$

$$\frac{dP_0^{(2)}(t)}{dt} = -(\lambda+\mu_2)P_0^{(2)}(t) + r\int_0^\infty \mu_1(x)P_0^{(1)}(x,t)dx, \tag{2}$$

$$\frac{dP_n^{(2)}(t)}{dt} = -(\lambda+\mu_2)P_n^{(2)}(t) + r\int_0^\infty \mu_1(x)P_n^{(1)}(x,t)dx + \lambda P_{n-1}^{(2)}(t),\ n \geq 1, \tag{3}$$

$$\frac{\partial P_0^{(1)}(x,t)}{\partial t} + \frac{\partial P_0^{(1)}(x,t)}{\partial x} = -(\lambda+\mu_1(x))P_0^{(1)}(x,t), \tag{4}$$

$$\frac{\partial P_n^{(1)}(x,t)}{\partial t} + \frac{\partial P_n^{(1)}(x,t)}{\partial x} = -(\lambda+\mu_1(x))P_n^{(1)}(x,t) + \lambda P_{n-1}^{(1)}(x,t),\ n \geq 1, \tag{5}$$

$$P_0^{(1)}(0,t) = \mu_2 P_1^{(2)}(t) + (1-r)\int_0^\infty \mu_1(x)P_1^{(1)}(x,t)dx + \lambda Q(t), \tag{6}$$

$$P_n^{(1)}(0,t) = \mu_2 P_{n+1}^{(2)}(t) + (1-r)\int_0^\infty \mu_1(x)P_{n+1}^{(1)}(x,t)dx, \quad n \geq 1, \tag{7}$$

$$Q(0) = 1, \qquad P_n^{(2)}(0) = 0, \qquad P_n^{(1)}(x,0) = 0, \quad n \geq 0. \tag{8}$$

Here $Q(t)$ represents the probability that at time $t$, there is no customer in the system and the server is idle. $P_n^{(1)}(x,t)$ represents the probability that at time $t$, there are $n$ customers in the queue [not system (queue + service)] excluding the one being provided the first essential service and elapsed service time of this customer is $x$. $P_n^{(2)}(t)$ represents the probability that at time $t$, there are $n$ customers in the queue excluding one customer being provided the second optional service. $\lambda$ is the arrival rate of customer. $\mu_1(x)$ is the service rate of completion of the first essential service. $\mu_2$ is the service rate of completion of the second optional service. $r$ is the probability that customer may opt to accept the second service.

We choose the state space as follows.

$$X = \left\{ \ g = (P^{(1)}, P^{(2)}) \ \middle| \ \|g\| = \|P^{(1)}\| + \|P^{(2)}\| < \infty \ \right\},$$

where

$$P^{(1)} = (Q, P_0^{(1)}(x), P_1^{(1)}(x), P_2^{(1)}(x), \cdots) \in \mathbb{R} \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots,$$

$$P^{(2)} = (P_0^{(2)}, P_1^{(2)}, P_2^{(2)}, P_3^{(2)}, \cdots\cdots) \in l^1,$$

$$\|P^{(1)}\| = |Q| + \sum_{n=0}^{\infty} \|P_n^{(1)}\|_{L^1[0,\infty)}, \qquad \|P^{(2)}\| = \sum_{n=0}^{\infty} |P_n^{(2)}|.$$

It is obvious that $X$ is a Banach space. For simplicity, we denote

$$A(P^{(1)}, P^{(2)})(x) = \left( \begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} Q \\ P_0^{(1)}(x) \\ P_1^{(1)}(x) \\ \vdots \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} P_0^{(2)} \\ P_1^{(2)} \\ P_2^{(2)} \\ \vdots \end{pmatrix} \right),$$

$$D(A) = \Big\{ (P^{(1)}, P^{(2)}) \in X \mid \frac{dP_n^{(1)}}{dx} \in L^1[0,\infty), P_n^{(1)}(x) \text{is absolutely continuous},$$
$$n \geq 0\,, \quad P^{(1)}(0) = \int_0^\infty \Gamma_1 P^{(1)}(x)dx + \int_0^\infty \Gamma_2 P^{(2)} dx,$$
$$|Q| + \sum_{n=0}^{\infty} \|\frac{dP_n^{(1)}}{dx}\|_{L^1[0,\infty)} + \sum_{n=0}^{\infty} |P_n^{(2)}| < \infty \Big\},$$

$$U(P^{(1)}, P^{(2)})(x)$$
$$= \left( \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & -(\lambda+\mu_1(x)) & 0 & \cdots \\ 0 & \lambda & -(\lambda+\mu_1(x)) & \cdots \\ 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} Q \\ P_0^{(1)}(x) \\ P_1^{(1)}(x) \\ P_2^{(1)}(x) \\ \vdots \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -(\lambda+\mu_2) & 0 & 0 & \cdots \\ \lambda & -(\lambda+\mu_2) & 0 & \cdots \\ 0 & \lambda & -(\lambda+\mu_2) & \cdots \\ 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} P_0^{(2)} \\ P_1^{(2)} \\ P_2^{(2)} \\ P_3^{(2)} \\ \vdots \end{pmatrix} \right),$$

$$E(P^{(1)},P^{(2)})(x) = \left( \begin{pmatrix} \mu_2 P_0^{(2)} + (1-r)\int_0^\infty \mu_1(x) P_0^{(1)}(x)dx \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} r\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx \\ r\int_0^\infty \mu_1(x)P_1^{(1)}(x)dx \\ r\int_0^\infty \mu_1(x)P_2^{(1)}(x)dx \\ \vdots \end{pmatrix} \right),$$

$$D(U) = X, \quad D(E) = X\,,$$

$$\Gamma_1 = \begin{pmatrix} e^{-x} & 0 & 0 & 0 & \cdots \\ \lambda e^{-x} & 0 & (1-r)\mu_1(x) & 0 & \cdots \\ 0 & 0 & 0 & (1-r)\mu_1(x) & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

$$\Gamma_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & \mu_2 e^{-x} & 0 & \cdots \\ 0 & 0 & \mu_2 e^{-x} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Then the above system of equations (1)-(8) can be written as an abstract Cauchy problem in the Banach space $X$:

$$\left.\begin{array}{l} \dfrac{d(P^{(1)},P^{(2)})(t)}{dt} = (A+U+E)(P^{(1)},P^{(2)})(t), t\in[0,\infty) \\ (P^{(1)},P^{(2)})(0) = (P^{(1)}(0),P^{(2)}(0)), \\ P^{(1)}(0) = (1,0,0,0,\cdots), P^{(2)}(0) = (0,0,0,0,\cdots) \end{array}\right\}. \tag{9}$$

Throughout this paper, we assume that $\mu = \sup\limits_{x\in[0,\infty)} \mu_1(x) < \infty$.

## 2. Main Results

**Theorem 1.** *$A$+$U$+$E$ generates a positive contraction $C_0$-semigroup $T(t)$.*

*Proof.* We split the proof of this theorem into four steps. First we prove that $(\gamma I - A)^{-1}$ exists and $\|(\gamma I - A)^{-1}\| < \frac{1}{\gamma-(\mu(1-r)+\mu_2)}$ when $\gamma > \mu(1-r)+\mu_2$.

Second we prove that $D(A)$ is dense in $X$. Thus by combining the above two steps with the Hille-Yosida Theorem we deduce that $A$ generates a $C_0$-semigroup. Third we show that $U$ and $E$ are bounded linear operators, from which together with the first and second steps, we derive that $A+U+E$ generates a $C_0$-semigroup. Fourth we prove that $A+U+E$ is a dispersive operator. Thus by using the Phillips Theorem we obtain the desired result.

For any given $(y^{(1)}, y^{(2)}) \in X$, consider the equation $(\gamma I - A)(P^{(1)}, P^{(2)}) = (y^{(1)}, y^{(2)})$. It is equivalent to the following equations:

$$(\gamma+\lambda)Q = y_0, \tag{10}$$

$$\gamma P_n^{(2)} = y_n^{(2)}, \quad n \geq 0, \tag{11}$$

$$\frac{dP_n^{(1)}(x)}{dx} = -\gamma P_n^{(1)}(x) + y_n^{(1)}, \quad n \geq 0, \tag{12}$$

$$P_0^{(1)}(0) = (1-r)\int_0^\infty \mu_1(x)P_1^{(1)}(x)dx + \mu_2 P_1^{(2)} + \lambda Q, \tag{13}$$

$$P_n^{(1)}(0) = (1-r)\int_0^\infty \mu_1(x)P_{n+1}^{(1)}(x)dx + \mu_2 P_{n+1}^{(2)}, \quad n \geq 1. \tag{14}$$

Solving (10), (11), (12) we have

$$Q = \frac{1}{\gamma+\lambda}y_0, \tag{15}$$

$$P_n^{(2)} = \frac{1}{\gamma}y_n^{(2)}, \quad n \geq 0 \tag{16}$$

$$P_n^{(1)}(x) = d_n e^{-\gamma x} + e^{-\gamma x}\int_0^x y_n^{(1)}(\tau)e^{\gamma\tau}d\tau. \tag{17}$$

From (13), (14), (15), (16) and (17) we deduce

$$\begin{aligned} d_0 = P_0^{(1)}(0) &= (1-r)\int_0^\infty \mu_1(x)P_1^{(1)}(x)dx + \mu_2 P_1^{(2)} + \lambda Q \\ &= (1-r)\int_0^\infty \left[d_1 e^{-\gamma x} + e^{-\gamma x}\int_0^x y_1^{(1)}(\tau)e^{\gamma\tau}d\tau\right]\mu_1(x)dx + \mu_2 P_1^{(2)} + \lambda Q \\ &= d_1(1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}dx + \frac{\lambda}{\gamma+\lambda}y_0 \\ &\quad + (1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_1^{(1)}(\tau)e^{\gamma\tau}d\tau dx + \frac{\mu_2}{\gamma}y_1^{(2)}, \end{aligned} \tag{18}$$

$$\begin{aligned} d_n = P_n^{(1)}(0) &= (1-r)\int_0^\infty \mu_1(x)P_{n+1}^{(1)}(x)dx + \mu_2 P_{n+1}^{(2)} \\ &= (1-r)\int_0^\infty \left[d_{n+1}e^{-\gamma x} + e^{-\gamma x}\int_0^x y_{n+1}^{(1)}(\tau)e^{\gamma\tau}d\tau\right]\mu_1(x)dx + \mu_2 P_{n+1}^{(2)} \end{aligned}$$

$$= d_{n+1}(1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}dx$$
$$+ (1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_{n+1}^{(1)}(\tau)e^{\gamma\tau}d\tau dx + \frac{\mu_2}{\gamma}y_{n+1}^{(2)}, \quad n\geq 1. \quad (19)$$

If we set

$$C = \begin{pmatrix} 1 & (r-1)\int_0^\infty \mu_1(x)e^{-\gamma x}dx & 0 & 0 & \cdots \\ 0 & 1 & (r-1)\int_0^\infty \mu_1(x)e^{-\gamma x}dx & 0 & \cdots \\ 0 & 0 & 1 & (r-1)\int_0^\infty \mu_1(x)e^{-\gamma x}dx & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad \vec{d} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ \vdots \end{pmatrix},$$

then from this together with (18) and (19) we have

$$C\vec{d} = \begin{pmatrix} (1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_1^{(1)}(\tau)e^{\gamma\tau}d\tau dx + \frac{\mu_2}{\gamma}y_1^{(2)} + \frac{\lambda}{\gamma+\lambda}y_0 \\ (1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_2^{(1)}(\tau)e^{\gamma\tau}d\tau dx + \frac{\mu_2}{\gamma}y_2^{(2)} \\ (1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_3^{(1)}(\tau)e^{\gamma\tau}d\tau dx + \frac{\mu_2}{\gamma}y_3^{(2)} \\ (1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_4^{(1)}(\tau)e^{\gamma\tau}d\tau dx + \frac{\mu_2}{\gamma}y_4^{(2)} \\ \vdots \end{pmatrix}, \quad (20)$$

Similar to [3], it is easy to calculate

$$C^{-1} = \begin{pmatrix} 1 & (1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}dx & \left((1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}dx\right)^2 & \left((1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}dx\right)^3 & \cdots \\ 0 & 1 & (1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}dx & \left((1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}dx\right)^2 & \cdots \\ 0 & 0 & 1 & (1-r)\int_0^\infty \mu_1(x)e^{-\gamma x}dx & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

From which together with (20) it follows that

$$d_0 = \frac{\lambda}{\gamma+\lambda} y_0 + \frac{\mu_2}{\gamma} \sum_{k=0}^{\infty} \left( \int_0^\infty \mu_1(x) e^{-\gamma x} dx \right)^k y_{k+1}^{(2)}$$
$$+ (1-r) \sum_{k=0}^{\infty} \left( (1-r) \int_0^\infty \mu_1(x) e^{-\gamma x} dx \right)^k \int_0^\infty \mu_1(x) e^{-\gamma x} \int_0^x y_{k+1}^{(1)}(\tau)$$
$$\times e^{\gamma \tau} d\tau dx, \quad (21)$$

$$d_n = \frac{\mu_2}{\gamma} \sum_{k=0}^{\infty} \left( (1-r) \int_0^\infty \mu_1(x) e^{-\gamma x} dx \right)^k y_{n+k+1}^{(2)}$$
$$+ (1-r) \sum_{k=0}^{\infty} \left( (1-r) \int_0^\infty \mu_1(x) e^{-\gamma x} dx \right)^k \int_0^\infty \mu_1(x) e^{-\gamma x} \int_0^x y_{n+k+1}^{(1)}(\tau)$$
$$\times e^{\gamma \tau} d\tau dx, \quad n \geq 1. \quad (22)$$

By using (15), (17) and the Fubini Theorem we have

$$\|P_n^{(1)}\|_{L^1[0,\infty)} = \int_0^\infty |P_n^{(1)}(x)| dx = \int_0^\infty \left| d_n e^{-\gamma x} + e^{-\gamma x} \int_0^x y_n^{(1)}(\tau) e^{-\gamma \tau} d\tau \right| dx$$
$$\leq |d_n| \int_0^\infty e^{-\gamma x} dx + \int_0^\infty e^{-\gamma x} \int_0^x |y_n^{(1)}(\tau)| e^{-\gamma \tau} d\tau dx$$
$$= |d_n| \int_0^\infty e^{-\gamma x} dx + \int_0^\infty |y_n^{(1)}(\tau)| e^{-\gamma \tau} \int_\tau^\infty e^{-\gamma x} dx d\tau$$
$$= \frac{1}{\gamma} |d_n| + \frac{1}{\gamma} \|y_n^{(1)}\|_{L^1[0,\infty)} \quad (23)$$

$$\Longrightarrow \|P^{(1)}\| = |Q| + \sum_{n=0}^{\infty} \|P_n^{(1)}\|_{L^1[0,\infty)}$$
$$\leq \frac{1}{\gamma+\lambda} |y_0| + \frac{1}{\gamma} \sum_{n=0}^{\infty} |d_n| + \frac{1}{\gamma} \sum_{n=0}^{\infty} \|y_n^{(1)}\|_{L^1[0,\infty)}. \quad (24)$$

Combining (21) and (22) with the Fubini Theorem we estimate

$$\sum_{n=0}^{\infty} |d_n| \leq \frac{\lambda}{\gamma+\lambda} |y_0| + \frac{\mu_2}{\gamma} \sum_{k=0}^{\infty} \left( (1-r) \int_0^\infty \mu_1(x) e^{-\gamma x} dx \right)^k |y_{k+1}^{(2)}|$$
$$+ (1-r) \sum_{k=0}^{\infty} \left( (1-r) \int_0^\infty \mu_1(x) e^{-\gamma x} dx \right)^k \int_0^\infty \mu_1(x) e^{-\gamma x} \int_0^x |y_{k+1}^{(1)}(\tau)| e^{\gamma \tau} d\tau dx$$
$$+ \frac{\mu_2}{\gamma} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left( (1-r) \int_0^\infty \mu_1(x) e^{-\gamma x} dx \right)^k |y_{n+k+1}^{(2)}|$$

$$+(1-r)\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left((1-r)\int_0^{\infty}\mu_1(x)e^{-\gamma x}dx\right)^k\int_0^{\infty}\mu_1(x)e^{-\gamma x}\int_0^x|y_{n+k+1}^{(1)}(\tau)|e^{\gamma\tau}d\tau dx$$

$$=\frac{\lambda}{\gamma+\lambda}|y_0|+\frac{\mu_2}{\gamma}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left((1-r)\int_0^{\infty}\mu_1(x)e^{-\gamma x}dx\right)^k|y_{n+k+1}^{(2)}|$$

$$+(1-r)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left((1-r)\int_0^{\infty}\mu_1(x)e^{-\gamma x}dx\right)^k\int_0^{\infty}\mu_1(x)e^{-\gamma x}\int_0^x|y_{n+k+1}^{(1)}(\tau)|e^{\gamma\tau}d\tau dx$$

$$=\frac{\lambda}{\gamma+\lambda}|y_0|+\frac{\mu_2}{\gamma}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left((1-r)\int_0^{\infty}\mu_1(x)e^{-\gamma x}dx\right)^k|y_{n+k+1}^{(2)}|$$

$$+(1-r)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left((1-r)\int_0^{\infty}\mu_1(x)e^{-\gamma x}dx\right)^k\int_0^{\infty}|y_{n+k+1}^{(1)}(\tau)|e^{\gamma\tau}\int_{\tau}^{\infty}\mu_1(x)e^{-\gamma x}dxd\tau$$

$$\leq\frac{\lambda}{\gamma+\lambda}|y_0|+\frac{\mu_2}{\gamma}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left((1-r)\mu\int_0^{\infty}e^{-\gamma x}dx\right)^k|y_{n+k+1}^{(2)}|$$

$$+\mu(1-r)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left((1-r)\mu\int_0^{\infty}e^{-\gamma x}dx\right)^k\int_0^{\infty}|y_{n+k+1}^{(1)}(\tau)|e^{\gamma\tau}\int_{\tau}^{\infty}e^{-\gamma x}dxd\tau=$$

$$\frac{\lambda}{\gamma+\lambda}|y_0|+\frac{\mu_2}{\gamma}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{\mu(1-r)}{\gamma}\right)^k|y_{n+k+1}^{(2)}|+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{\mu(1-r)}{\gamma}\right)^{k+1}\int_0^{\infty}|y_{n+k+1}^{(1)}(\tau)|d\tau$$

$$=\frac{\lambda}{\gamma+\lambda}|y_0|+\frac{\mu_2}{\gamma}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{\mu(1-r)}{\gamma}\right)^k|y_{n+k+1}^{(2)}|+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{\mu(1-r)}{\gamma}\right)^{k+1}\|y_{n+k+1}^{(1)}\|_{L^1[0,\infty)}$$

$$=\frac{\lambda}{\gamma+\lambda}|y_0|+\frac{\mu_2}{\gamma}\sum_{k=0}^{\infty}\left(\frac{\mu(1-r)}{\gamma}\right)^k\sum_{n=0}^{\infty}|y_{n+k+1}^{(2)}|+\sum_{k=0}^{\infty}\left(\frac{\mu(1-r)}{\gamma}\right)^{k+1}\sum_{n=0}^{\infty}\|y_{n+k+1}^{(1)}\|_{L^1[0,\infty)}$$

$$\leq\frac{\lambda}{\gamma+\lambda}|y_0|+\frac{\mu_2}{\gamma}\sum_{k=0}^{\infty}\left(\frac{\mu(1-r)}{\gamma}\right)^k\sum_{n=0}^{\infty}|y_n^{(2)}|+\sum_{k=0}^{\infty}\left(\frac{\mu(1-r)}{\gamma}\right)^{k+1}\sum_{n=0}^{\infty}\|y_n^{(1)}\|_{L^1[0,\infty)}$$

$$=\frac{\lambda}{\gamma+\lambda}|y_0|+\frac{\mu_2}{\gamma-\mu(1-r)}\sum_{n=0}^{\infty}|y_n^{(2)}|+\frac{\mu(1-r)}{\gamma-\mu(1-r)}\sum_{n=0}^{\infty}\|y_n^{(1)}\|_{L^1[0,\infty)}. \quad (25)$$

From (16) it is easy to derive

$$\|P^{(2)}\|=\sum_{n=0}^{\infty}|P_n^{(2)}|=\frac{1}{\gamma}\sum_{n=0}^{\infty}|y_n^{(2)}|. \quad (26)$$

By (24), (25) and (26) we get without loss of generality, assume $r>\frac{\mu^2(1-r)^2-\mu_2^2}{\mu(1-r)}$,

$$\|(P^{(1)},P^{(2)})\|=\|P^{(1)}\|+\|P^{(2)}\|$$

$$\leq\frac{1}{\gamma+\lambda}|y_0|+\frac{1}{\gamma}\sum_{n=0}^{\infty}|d_n|+\frac{1}{\gamma}\sum_{n=0}^{\infty}\|y_n^{(1)}\|_{L^1[0,\infty)}+\frac{1}{\gamma}\sum_{n=0}^{\infty}|y_n^{(2)}|$$

$$\leq \frac{1}{\gamma+\lambda}|y_0| + \frac{\lambda}{\gamma(\gamma+\lambda)}|y_0| + \frac{\mu_2}{\gamma(\gamma-\mu(1-r))}\sum_{n=0}^{\infty}|y_n^{(2)}|$$

$$+ \frac{\mu(1-r)}{\gamma(\gamma-\mu(1-r))}\sum_{n=0}^{\infty}\|y_n^{(1)}\|_{L^1[0,\infty)} + \frac{1}{\gamma}\sum_{n=0}^{\infty}\|y_n^{(1)}\|_{L^1[0,\infty)} + \frac{1}{\gamma}\sum_{n=0}^{\infty}|y_n^{(2)}|$$

$$= \frac{1}{\gamma}|y_0| + \left(\frac{1}{\gamma} + \frac{\mu(1-r)}{\gamma(\gamma-\mu(1-r))}\right)\sum_{n=0}^{\infty}\|y_n^{(1)}\|_{L^1[0,\infty)} + \left(\frac{1}{\gamma} + \frac{\mu_2}{\gamma(\gamma-\mu(1-r))}\right)$$

$$\times\sum_{n=0}^{\infty}|y_n^{(2)}| \leq \frac{1}{\gamma}|y_0| + \frac{1}{\gamma-\mu(1-r)}\sum_{n=0}^{\infty}\|y_n^{(1)}\|_{L^1[0,\infty)} + \frac{\gamma-\mu(1-r)+\mu_2}{\gamma(\gamma-\mu(1-r))}\sum_{n=0}^{\infty}|y_n^{(2)}|$$

$$< \frac{1}{\gamma-\mu(1-r)-\mu_2}\left\{|y_0| + \sum_{n=0}^{\infty}\|y_n^{(1)}\|_{L^1[0,\infty)} + \sum_{n=0}^{\infty}|y_n^{(2)}|\right\}$$

$$= \frac{1}{\gamma-\mu(1-r)-\mu_2}\|(y^{(1)}, y^{(2)})\|. \quad (27)$$

(27) shows that when $\gamma > \mu(1-r)+\mu_2$, $(\gamma I - A)^{-1} : X \mapsto X$ exists and satisfies $\|(\gamma I - A)^{-1}\| < \frac{1}{\gamma-(\mu(1-r)+\mu_2)}$.

Second, we prove that $D(A)$ is dense in $X$.

As far as the second step is concerned, from $|Q| + \sum_{n=0}^{\infty}\|P_n^{(1)}\|_{L^1[0,\infty)} + \sum_{n=0}^{\infty}|P_n^{(2)}| < \infty$ for $(P^{(1)}, P^{(2)}) \in X$ it follows that, for any $\varepsilon > 0$ there exists a positive integer $j$ such that $\sum_{n=j}^{\infty}\|P_n^{(1)}\|_{L^1[0,\infty)} + \sum_{n=j}^{\infty}|P_n^{(2)}| < \varepsilon$. Let

$$L = \left\{(P^{(1)}, P^{(2)}) \,\middle|\, \begin{array}{l} P_i^{(1)} \in L^1[0,\infty), P_i^{(2)} \in l^1, i = 0, 1, \cdots, k, \\ k \text{ is a finite positive integer.} \end{array}\right\}.$$

Here $(P^{(1)}, P^{(2)}) \in L$ means that the prior finite components of $P^{(1)}$ and $P^{(2)}$ are nonzero, others are zero, then it is obvious that $L$ is dense in $X$. If we set

$$Z = \left\{(P^{(1)}, P^{(2)}) \,\middle|\, \begin{array}{l} P_i^{(1)} \in C_0^{\infty}[0,\infty), \text{there exists } c_i > 0, \text{such that} \\ P_i^{(1)}(x) = 0,\ x \in [0, c_i],\ i = 0, 1, \cdots, N. \end{array}\right\},$$

then by [1], $Z$ is dense in $L$. From the above discussion we know that, in order to prove $D(A)$ is dense in $X$, it is sufficient to prove that $D(A)$ is dense in $Z$.

Take any $(P^{(1)}, P^{(2)}) \in Z$, there are a finite positive integer $N$ and number $c_i > 0$ such that $P^{(1)}(x) = (Q, P_0^{(1)}(x), \cdots, P_N^{(1)}(x), 0, \cdots)$, $P_i^{(1)}(x) = 0, x \in [0, c_i]$, $i = 0, 1, 2, \cdots, N$, $P^{(2)} = (P_0^{(2)}, P_1^{(2)}, \cdots, P_N^{(2)}, 0, 0, \cdots)$. This leads to

$P_i^{(1)}(x) = 0$ for $x \in [0, 2s]$, where $0 < 2s < \min\{c_0, c_1, \cdots, c_N\}$. Define

$$f^s(0) = \begin{pmatrix} Q \\ f_0^s(0) \\ f_1^s(0) \\ \vdots \\ f_{N-1}^s(0) \\ f_N^s(0) \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} Q \\ (1-r)\int_{2s}^{\infty} \mu_1(x) P_1^{(1)}(x)dx + \mu_2 P_1^{(2)} + \lambda Q \\ (1-r)\int_{2s}^{\infty} \mu_1(x) P_2^{(1)}(x)dx + \mu_2 P_2^{(2)} \\ \vdots \\ (1-r)\int_{2s}^{\infty} \mu_1(x) P_N^{(1)}(x)dx + \mu_2 P_N^{(2)} \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

$$(f^s, g)(x) = \left( \begin{pmatrix} Q \\ f_0^s(x) \\ f_1^s(x) \\ \vdots \\ f_{N-1}^s(x) \\ f_N^s(x) \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} P_0^{(2)} \\ P_1^{(2)} \\ P_2^{(2)} \\ \vdots \\ P_N^{(2)} \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \right),$$

where

$$f_i^s(x) = \begin{cases} f_i^s(0)(1-\frac{x}{s})^2 & x \in [0, s], \\ -\beta_i (x-s)^2 (x-2s)^2 & x \in [s, 2s], \\ P_i^{(1)}(x) & x \in [2s, \infty), \end{cases}$$

$i = 0, 1, \cdots, N-1, \quad f_N^s(x) = P_N^{(1)}(x)$.

Here

$$\beta_i = \frac{f_i^s(0)\int_0^s (1-\frac{x}{s})^2 \mu_1(x)dx}{\int_s^{2s} (x-s)^2 (x-2s)^2 \mu_1(x)dx}, \quad i = 0, 1, \cdots, N-1.$$

Then it is easy to verify that $(f^s, g) \in D(A)$ and

$$\begin{aligned} \|(P^{(1)}, P^{(2)}) - (f^s, g)\| &= \sum_{i=0}^{N-1} \int_0^{\infty} |P_i^{(1)}(x) - f_i^s(x)| dx \\ &= \sum_{i=0}^{N-1} \int_0^s |f_i^s(0)(1-\frac{x}{s})^2| dx + \sum_{i=0}^{N-1} \int_s^{2s} |\beta_i| (x-s)^2 (x-2s)^2 dx \end{aligned}$$

$$= \sum_{i=0}^{N-1} |f_i^s(0)| \frac{s}{3} + \sum_{i=0}^{N-1} |\beta_i| \frac{s^5}{30} \longrightarrow 0 \ \text{ as } \ s \to 0.$$

Which shows that $D(A)$ is dense in $Z$. That is, $D(A)$ is dense in $X$.

By the above two steps and the Hille-Yosida Theorem, we conclude that $A$ generates a $C_0$-semigroup (see [2], [3]).

Third we prove that $U$ and $E$ are bounded linear operators.

For any $(P^{(1)}, P^{(2)}) \in X$, by the definition of $U$ and $E$ we have

$$U(P^{(1)}, P^{(2)})(x) = \left( \begin{pmatrix} 0 \\ -(\lambda + \mu_1(x)) P_0^{(1)}(x) \\ -(\lambda + \mu_1(x)) P_1^{(1)}(x) + \lambda P_0^{(1)}(x) \\ -(\lambda + \mu_1(x)) P_2^{(1)}(x) + \lambda P_1^{(1)}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} -(\lambda + \mu_2) P_0^{(2)} \\ -(\lambda + \mu_2) P_1^{(2)} + \lambda P_0^{(2)} \\ -(\lambda + \mu_2) P_2^{(2)} + \lambda P_1^{(2)} \\ -(\lambda + \mu_2) P_3^{(2)} + \lambda P_2^{(2)} \\ \vdots \end{pmatrix} \right),$$

$$\begin{aligned} \|U(P^{(1)}, P^{(2)})\| &\le \sum_{n=0}^{\infty} \int_0^{\infty} (\lambda + \mu_1(x)) |P_n^{(1)}(x)| dx + \sum_{n=0}^{\infty} \int_0^{\infty} \lambda |P_n^{(1)}(x)| dx \\ &\quad + \sum_{n=0}^{\infty} (\lambda + \mu_2) |P_n^{(2)}| + \sum_{n=0}^{\infty} \lambda |P_n^{(2)}| \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} \mu_1(x) |P_n^{(1)}(x)| dx + 2\lambda \sum_{n=0}^{\infty} \int_0^{\infty} |P_n^{(1)}(x)| dx + \sum_{n=0}^{\infty} (2\lambda + \mu_2) |P_n^{(2)}| \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} \mu_1(x) |P_n^{(1)}(x)| dx + 2\lambda \sum_{n=0}^{\infty} \|P_n^{(1)}\|_{L^1[0,\infty)} + (2\lambda + \mu_2) \sum_{n=0}^{\infty} |P_n^{(2)}| \\ &\le \mu \sum_{n=0}^{\infty} \int_0^{\infty} |P_n^{(1)}(x)| dx + 2\lambda \sum_{n=0}^{\infty} \|P_n^{(1)}\|_{L^1[0,\infty)} + (2\lambda + \mu_2) \sum_{n=0}^{\infty} |P_n^{(2)}| \\ &= (\mu + 2\lambda) \sum_{n=0}^{\infty} \|P_n^{(1)}\|_{L^1[0,\infty)} + (2\lambda + \mu_2) \|P^{(2)}\| \\ &\le (\mu + 2\lambda) \|P^{(1)}\| + (\mu_2 + 2\lambda) \|P^{(2)}\| \le (\mu + \mu_2 + 2\lambda) \|(P^{(1)}, P^{(2)})\|. \quad (28) \end{aligned}$$

$E(P^{(1)}, P^{(2)})(x)$

$$= \left( \begin{pmatrix} (1-r)\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx + \mu_2 P_0^{(2)} \\ \vdots \\ \vdots \end{pmatrix}, \begin{pmatrix} r\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx \\ r\int_0^\infty \mu_1(x)P_1^{(1)}(x)dx \\ r\int_0^\infty \mu_1(x)P_2^{(1)}(x)dx \\ \vdots \end{pmatrix} \right),$$

$$\begin{aligned} &\|E(P^{(1)}, P^{(2)})\| \\ &\quad \le \mu_2|P_0^{(2)}| + (1-r)\int_0^\infty \mu_1(x)|P_0^{(1)}(x)|dx + r\sum_{n=0}^\infty \int_0^\infty \mu_1(x)|P_n^{(1)}(x)|dx \\ &\quad = \mu_2|P_0^{(2)}| + \int_0^\infty \mu_1(x)|P_0^{(1)}(x)|dx + r\sum_{n=1}^\infty \int_0^\infty \mu_1(x)|P_n^{(1)}(x)|dx \\ &\le \mu_2|P_0^{(2)}| + \sum_{n=0}^\infty \int_0^\infty \mu_1(x)|P_n^{(1)}(x)|dx \le \mu_2\|P^{(2)}\| + \mu\sum_{n=0}^\infty \int_0^\infty |P_n^{(1)}(x)|dx \\ &\quad = \mu_2\|P^{(2)}\| + \mu\|P^{(1)}\| \le (\mu+\mu_2)\|(P^{(1)}, P^{(2)})\|. \qquad (29) \end{aligned}$$

(28) and (29) show that $U$ and $E$ are bounded operators. Obviously, $U$ and $E$ are linear. Therefore, from the perturbation theory of $C_0$-semigroup (see [2], [3]) we know that $A+U+E$ generates a $C_0$-semigroup $T(t)$.

Fourth we prove $A+U+E$ is a dispersive operator.

For any $(P^{(1)}, P^{(2)}) \in D(A)$, we may choose

$$\phi(x) = \left( \begin{pmatrix} \frac{[Q]^+}{Q} \\ \frac{[P_0^{(1)}(x)]^+}{P_0^{(1)}(x)} \\ \frac{[P_1^{(1)}(x)]^+}{P_1^{(1)}(x)} \\ \vdots \end{pmatrix}, \begin{pmatrix} \frac{[P_0^{(2)}]^+}{P_0^{(2)}} \\ \frac{[P_1^{(2)}]^+}{P_1^{(2)}} \\ \frac{[P_2^{(2)}]^+}{P_2^{(2)}} \\ \vdots \end{pmatrix} \right),$$

where

$$[Q]^+ = \begin{cases} Q, & \text{if } \ Q \ge 0, \\ 0, & \text{if } \ Q < 0, \end{cases} \quad [P_n^{(2)}]^+ = \begin{cases} P_n^{(2)}, & \text{if } \ P_n^{(2)} \ge 0, \\ 0, & \text{if } \ P_n^{(2)} < 0, \end{cases}$$

$$[P_n^{(1)}(x)]^+ = \begin{cases} P_n^{(1)}(x), & \text{if } \ P_n^{(1)}(x) \ge 0, \\ 0, & \text{if } \ P_n^{(1)}(x) < 0, \end{cases} \quad n = 0, 1, 2, \cdots$$

Then by using the boundary conditions on $(P^{(1)}, P^{(2)})$ and the formula (4-57) in [3], p. 126, we have

$$\langle (A+U+E)(P^{(1)}, P^{(2)}),\ \phi \rangle$$

$$
\begin{aligned}
&= \frac{[Q]^+}{Q}\left(-\lambda Q + \mu_2 P_0^{(2)} + (1-r)\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx\right) \\
&+ \int_0^\infty \left(-\frac{dP_0^{(1)}(x)}{dx} - (\lambda+\mu_1(x))P_0^{(1)}(x)\right)\frac{[P_0^{(1)}(x)]^+}{P_0^{(1)}(x)}dx \\
&+ \sum_{n=1}^\infty \int_0^\infty \left[-\frac{dP_n^{(1)}(x)}{dx} - (\lambda+\mu_1(x))P_n^{(1)}(x) + \lambda P_{n-1}^{(1)}(x)\right]\frac{[P_n^{(1)}(x)]^+}{P_n^{(1)}(x)}dx \\
&+ \left(-(\lambda+\mu_2)P_0^{(2)} + r\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx\right)\frac{[P_0^{(2)}]^+}{P_0^{(2)}} \\
&+ \sum_{n=1}^\infty \left[-(\lambda+\mu_2)P_n^{(2)} + r\int_0^\infty \mu_1(x)P_n^{(1)}(x)dx + \lambda P_{n-1}^{(2)}\right]\frac{[P_n^{(2)}]^+}{P_n^{(2)}} \\
&= \frac{[Q]^+}{Q}\left(-\lambda Q + \mu_2 P_0^{(2)} + (1-r)\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx\right) \\
&- \sum_{n=0}^\infty \int_0^\infty \frac{dP_n^{(1)}(x)}{dx}\frac{[P_n^{(1)}(x)]^+}{P_n^{(1)}(x)} - \sum_{n=0}^\infty \int_0^\infty (\lambda+\mu_1(x))[P_n^{(1)}(x)]^+dx \\
&+ \lambda\sum_{n=1}^\infty \int_0^\infty P_{n-1}^{(1)}(x)\frac{[P_n^{(1)}(x)]^+}{P_n^{(1)}(x)}dx - \sum_{n=0}^\infty (\lambda+\mu_2)[P_n^{(2)}]^+ \\
&+ r\sum_{n=0}^\infty \frac{[P_n^{(2)}]^+}{P_n^{(2)}}\int_0^\infty \mu_1(x)P_n^{(1)}(x)dx + \lambda\sum_{n=1}^\infty P_{n-1}^{(2)}\frac{[P_n^{(2)}]^+}{P_n^{(2)}} \\
&= \frac{[Q]^+}{Q}\left(-\lambda Q + \mu_2 P_0^{(2)} + (1-r)\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx\right) \\
&+\sum_{n=0}^\infty [P_n^{(1)}(0)]^+ - \sum_{n=0}^\infty \int_0^\infty (\lambda+\mu_1(x))[P_n^{(1)}(x)]^+dx + \lambda\sum_{n=1}^\infty \int_0^\infty P_{n-1}^{(1)}(x)\frac{[P_n^{(1)}(x)]^+}{P_n^{(1)}(x)}dx \\
&- (\lambda+\mu_2)\sum_{n=0}^\infty [P_n^{(2)}]^+ + r\sum_{n=0}^\infty \frac{[P_n^{(2)}]^+}{P_n^{(2)}}\int_0^\infty \mu_1(x)P_n^{(1)}(x)dx + \lambda\sum_{n=1}^\infty P_{n-1}^{(2)}\frac{[P_n^{(2)}]^+}{P_n^{(2)}} \\
&\le -\lambda[Q]^+ + \frac{[Q]^+}{Q}\left(\mu_2 P_0^{(2)} + (1-r)\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx\right) \\
&+ \lambda[Q]^+ + \mu_2[P_1^{(2)}]^+ + (1-r)\int_0^\infty \mu_1(x)[P_1^{(1)}(x)]^+dx \\
&+ \mu_2\sum_{n=1}^\infty [P_{n+1}^{(2)}]^+ + (1-r)\sum_{n=1}^\infty \int_0^\infty \mu_1(x)[P_{n+1}^{(1)}(x)]^+dx
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda+\mu_1(x))[P_n^{(1)}(x)]^+dx+\lambda\sum_{n=1}^{\infty}\int_0^{\infty}P_{n-1}^{(1)}(x)\frac{[P_n^{(1)}(x)]^+}{P_n^{(1)}(x)}dx\\
&-(\lambda+\mu_2)\sum_{n=0}^{\infty}[P_n^{(2)}]^+ + r\sum_{n=0}^{\infty}\frac{[P_n^{(2)}]^+}{P_n^{(2)}}\int_0^{\infty}\mu_1(x)P_n^{(1)}(x)dx+\lambda\sum_{n=1}^{\infty}P_{n-1}^{(2)}\frac{[P_n^{(2)}]^+}{P_n^{(2)}}\\
&=\frac{[Q]^+}{Q}\left(\mu_2P_0^{(2)}+(1-r)\int_0^{\infty}\mu_1(x)P_0^{(1)}(x)dx\right)\\
&+\mu_2\sum_{n=1}^{\infty}[P_n^{(2)}]^+ +(1-r)\sum_{n=1}^{\infty}\int_0^{\infty}\mu_1(x)[P_n^{(1)}(x)]^+dx\\
&-\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda+\mu_1(x))[P_n^{(1)}(x)]^+dx+\lambda\sum_{n=1}^{\infty}\int_0^{\infty}P_{n-1}^{(1)}(x)\frac{[P_n^{(1)}(x)]^+}{P_n^{(1)}(x)}dx\\
&-(\lambda+\mu_2)\sum_{n=0}^{\infty}[P_n^{(2)}]^+ + r\sum_{n=0}^{\infty}\frac{[P_n^{(2)}]^+}{P_n^{(2)}}\int_0^{\infty}\mu_1(x)P_n^{(1)}(x)dx+\lambda\sum_{n=1}^{\infty}P_{n-1}^{(2)}\frac{[P_n^{(2)}]^+}{P_n^{(2)}}\\
&\le\mu_2[P_0^{(2)}]^+ +(1-r)\int_0^{\infty}\mu_1(x)[P_0^{(1)}(x)]^+dx\\
&+\mu_2\sum_{n=1}^{\infty}[P_n^{(2)}]^+ +(1-r)\sum_{n=1}^{\infty}\int_0^{\infty}\mu_1(x)[P_n^{(1)}(x)]^+dx\\
&-\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda+\mu_1(x))[P_n^{(1)}(x)]^+dx+\lambda\sum_{n=1}^{\infty}\int_0^{\infty}[P_{n-1}^{(1)}(x)]^+dx\\
&-(\lambda+\mu_2)\sum_{n=0}^{\infty}[P_n^{(2)}]^+ + r\sum_{n=0}^{\infty}\int_0^{\infty}\mu_1(x)[P_n^{(1)}(x)]^+dx+\lambda\sum_{n=1}^{\infty}[P_{n-1}^{(2)}]^+\\
&=\mu_2\sum_{n=0}^{\infty}[P_n^{(2)}]^+ +(1-r)\sum_{n=0}^{\infty}\int_0^{\infty}\mu_1(x)[P_n^{(1)}(x)]^+dx\\
&-\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda+\mu_1(x))[P_n^{(1)}(x)]^+dx+\lambda\sum_{n=0}^{\infty}\int_0^{\infty}[P_n^{(1)}(x)]^+dx\\
&-(\lambda+\mu_2)\sum_{n=0}^{\infty}[P_n^{(2)}]^+ + r\sum_{n=0}^{\infty}\int_0^{\infty}\mu_1(x)[P_n^{(1)}(x)]^+dx+\lambda\sum_{n=0}^{\infty}[P_n^{(2)}]^+ = 0. \quad (30)
\end{aligned}
$$

By combining the first step, the second step, (30) and the Phillips Theorem (see [3], [5]) we know that $A+U+E$ generates a positive contraction $C_0$-semigroup. By uniqueness of $C_0$-semigroup (see [3], [5]) we know that this positive contraction $C_0$-semigroup is just $T(t)$. Thus we complete the proof of Theorem 1. □

If we take a set $Y$ in $X$ as

$$Y=\left\{g\in X\;\middle|\;\begin{array}{l}g(x)=(P^{(1)},P^{(2)}),x\in[0,\infty),Q\geq 0,\\ P_n^{(1)}(x)\geq 0,P_n^{(2)}\geq 0,\quad n=0,1,2,\cdots\end{array}\right\},$$

then $Y$ is a cone in $X$. It is easy to verify that the dual space $X^*$ of $X$ is

$$X^*=\Big\{g^*=(P^{(1)*},P^{(2)*})\mid P^{(1)*}\in\mathbb{R}\times L^\infty[0,\infty)\times L^\infty[0,\infty)\times\cdots,P^{(2)*}\in l^\infty,$$

$$|||g^*|||=\max\left\{|Q^*|,\sup_{n\geq 0}|P_n^{(2)*}|,\sup_{n\geq 0}\|P_n^{(1)*}\|_{L^\infty[0,\infty)}\right\}<\infty\Big\}.$$

Obviously, $X^*$ is a Banach space. If $g\in D(A)$, then we take

$$g^*(x)=\|g\|\left(\begin{pmatrix}1\\1\\1\\\vdots\end{pmatrix},\begin{pmatrix}1\\1\\1\\\vdots\end{pmatrix}\right)\in X^*.$$

For $g^*\in X^*$ and $g\in Y$, by using boundary conditions we have

$$\begin{aligned}
\langle(A+U+E)g,\ g^*\rangle&=\left(-\lambda Q+\mu_2P_0^{(2)}+(1-r)\int_0^\infty\mu_1(x)P_0^{(1)}(x)dx\right)\|g\|\\
&+\int_0^\infty\left(-\frac{dP_0^{(1)}(x)}{dx}-(\lambda+\mu_1(x))P_0^{(1)}(x)\right)\|g\|dx\\
&+\sum_{n=1}^\infty\int_0^\infty\left(-\frac{dP_n^{(1)}(x)}{dx}-(\lambda+\mu_1(x))P_n^{(1)}(x)+\lambda P_{n-1}^{(1)}(x)\right)\|g\|dx\\
&+\left(-(\lambda+\mu_2)P_0^{(2)}+r\int_0^\infty\mu_1(x)P_0^{(1)}(x)dx\right)\|g\|\\
&+\sum_{n=1}^\infty\left(-(\lambda+\mu_2)P_n^{(2)}+r\int_0^\infty\mu_1(x)P_n^{(1)}(x)dx+\lambda P_{n-1}^{(2)}\right)\|g\|\\
&=\Big\{\left(-\lambda Q+\mu_2P_0^{(2)}+(1-r)\int_0^\infty\mu_1(x)P_0^{(1)}(x)dx\right)\\
&-\sum_{n=0}^\infty\int_0^\infty\frac{dP_n^{(1)}(x)}{dx}dx-\sum_{n=0}^\infty\int_0^\infty(\lambda+\mu_1(x))P_n^{(1)}(x)dx+\lambda\sum_{n=1}^\infty\int_0^\infty P_{n-1}^{(1)}(x)dx\\
&-\sum_{n=0}^\infty(\lambda+\mu_2)P_n^{(2)}+r\sum_{n=0}^\infty\int_0^\infty\mu_1(x)P_n^{(1)}(x)dx+\lambda\sum_{n=1}^\infty P_{n-1}^{(2)}\Big\}\|g\|\\
&=\Big\{-\lambda Q+\mu_2P_0^{(2)}+(1-r)\int_0^\infty\mu_1(x)P_0^{(1)}(x)dx
\end{aligned}$$

$$
-\sum_{n=0}^{\infty}\int_0^{\infty} dP_n^{(1)}(x) - \sum_{n=0}^{\infty}\int_0^{\infty}(\lambda+\mu_1(x))P_n^{(1)}(x)dx + \lambda\sum_{n=0}^{\infty}\int_0^{\infty} P_n^{(1)}(x)dx
$$

$$
-\sum_{n=0}^{\infty}(\lambda+\mu_2)P_n^{(2)} + r\sum_{n=0}^{\infty}\int_0^{\infty}\mu_1(x)P_n^{(1)}(x)dx + \lambda\sum_{n=0}^{\infty}P_n^{(2)}\Big\}\|g\|
$$

$$
=\Big\{-\lambda Q + \mu_2 P_0^{(2)} + (1-r)\int_0^{\infty}\mu_1(x)P_0^{(1)}(x)dx
$$

$$
+\sum_{n=0}^{\infty}P_n^{(1)}(0) - \sum_{n=0}^{\infty}\int_0^{\infty}(\lambda+\mu_1(x))P_n^{(1)}(x)dx + \lambda\sum_{n=0}^{\infty}\int_0^{\infty}P_n^{(1)}(x)dx
$$

$$
-\sum_{n=0}^{\infty}(\lambda+\mu_2)P_n^{(2)} + r\sum_{n=0}^{\infty}\int_0^{\infty}\mu_1(x)P_n^{(1)}(x)dx + \lambda\sum_{n=0}^{\infty}P_n^{(2)}\Big\}\|g\|
$$

$$
=\Big\{-\lambda Q + \mu_2 P_0^{(2)} + (1-r)\int_0^{\infty}\mu_1(x)P_0^{(1)}(x)dx
$$

$$
+\lambda Q + \mu_2 P_1^{(2)} + (1-r)\int_0^{\infty}\mu_1(x)P_1^{(1)}(x)dx
$$

$$
+\mu_2\sum_{n=1}^{\infty}P_{n+1}^{(2)} + (1-r)\sum_{n=1}^{\infty}\int_0^{\infty}\mu_1(x)P_{n+1}^{(1)}(x)dx
$$

$$
-\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda+\mu_1(x))P_n^{(1)}(x)dx + \lambda\sum_{n=0}^{\infty}\int_0^{\infty}P_n^{(1)}(x)dx
$$

$$
-\sum_{n=0}^{\infty}(\lambda+\mu_2)P_n^{(2)} + r\sum_{n=0}^{\infty}\int_0^{\infty}\mu_1(x)P_n^{(1)}(x)dx + \lambda\sum_{n=0}^{\infty}P_n^{(2)}\Big\}\|g\|
$$

$$
=\Big\{\mu_2\sum_{n=0}^{\infty}P_n^{(2)} + (1-r)\sum_{n=0}^{\infty}\int_0^{\infty}\mu_1(x)P_n^{(1)}(x)dx + \lambda\sum_{n=0}^{\infty}P_n^{(2)}
$$

$$
+(r-1)\sum_{n=0}^{\infty}\int_0^{\infty}\mu_1(x)P_n^{(1)}(x)dx - \sum_{n=0}^{\infty}(\lambda+\mu_2)P_n^{(2)}\Big\}\|g\| = 0. \quad (31)
$$

(31) shows that $A+U+E$ is a conservative operator (see [2, 3]). Because $(P^{(1)},(P^{(2)})(0) \in D(A^2)$, by the result of Fattorini (see [2, 3]) we deduce the following result.

**Theorem 2.** *For the initial condition* $(P^{(1)},P^{(2)})(0) = (P^{(1)}(0),P^{(2)}(0))$, $P^{(1)}(0) = (1,0,0,0,\cdots)$, $P^{(2)}(0) = (0,0,0,0,\cdots)$, *we have*

$$
\|T(t)(P^{(1)},P^{(2)})(0)\| = \|(P^{(1)},P^{(2)})(0)\|, \quad t\in[0,\infty). \quad (32)
$$

Combining Theorem 1 with Theorem 2 we obtain the main result of this paper.

**Theorem 3.** *The system (9) has a unique positive time-dependent solution* $(P^{(1)}, P^{(2)})(x,t)$ *which satisfies*

$$\|(P^{(1)}, P^{(2)})(\cdot,t)\| = 1, \forall\ t \in [0,\infty). \tag{33}$$

*Proof.* From Theorem 1 and [3], [5] we know that the system (9) has a unique positive time-dependent solution $(P^{(1)}, P^{(2)})(x,t)$ which can be expressed as

$$(P^{(1)}, P^{(2)})(x,t) = T(t)(P^{(1)}, P^{(2)})(0). \tag{34}$$

By (32) and (34) we derive

$$\|(P^{(1)}, P^{(2)})(\cdot,t)\| = \|T(t)(P^{(1)}, P^{(2)})(0)\| = \|(P^{(1)}, P^{(2)})(0)\| = 1, \quad \forall\ t \in [0,\infty).$$

We complete the proof of Theorem 3. □

(33) just reflects the physical meaning of $(P^{(1)}, P^{(2)})(x,t)$.

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