International Journal of Pure and Applied Mathematics
Volume 50 No. 4 2009, 559-563

A PROOF OF HADAMARD INEQUALITY FOR NONSINGULAR OR POSITIVE DEFINITE MATRICES

Sumitra Purkayastha
Bayesian and Interdisciplinary Research Unit
Indian Statistical Institute
203, B.T. Road, Kolkata, 700 108, INDIA
e-mail: sumitra@isical.ac.in

Abstract: Hadamard inequality for nonsingular matrices states the following: if $\mathbf{A} = ((a_{ij}))$ is an $n \times n$ nonsingular matrix, then $[\det(\mathbf{A})]^2 \leq \prod_{i=1}^{n}(\sum_{j=1}^{n} a_{ij}^2)$. This result is equivalent to the following fact about positive definite matrices: if $\mathbf{B} = ((b_{ij}))$ is an $n \times n$ positive definite matrix, then $\det(\mathbf{B}) \leq \prod_{i=1}^{n} b_{ii}$. We provide a simple proof of this later fact that uses in a straightforward manner spectral decomposition of a positive definite matrix and the inequality between arithmetic mean and geometric mean of a set of positive numbers.

AMS Subject Classification: 15A45
Key Words: Hadamard inequality, positive definite matrix

1. Introduction

We begin by stating Hadamard inequality for nonsingular matrices.

Theorem 1. *Let* $\mathbf{A} = ((a_{ij}))$ *be an* $n \times n$ *nonsingular matrix. Then*

$$[\det(\mathbf{A})]^2 \leq \prod_{i=1}^{n}(\sum_{j=1}^{n} a_{ij}^2). \tag{1}$$

Remark 1. Notice that any singular matrix of order $n \times n$ can be written as the limit of a sequence of $n \times n$ non-singular matrices, where limit of a sequence

Received: November 11, 2008

of matrices $\mathbf{M}_n = ((m_{ij}^{(n)}))$ is defined as the matrix $\mathbf{M} \stackrel{def}{=} ((\lim_{n\to\infty} m_{ij}^{(n)}))$, assuming $\lim_{n\to\infty} m_{ij}^{(n)}$ to exist for every (i,j). Notice, moreover, that both sides of (1) are continuous functions of the entries of $\mathbf{A}$. Theorem 1 is, therefore, easily seen to hold *also* for any singular matrix of order $n \times n$.

Remark 2. Notice that a positive definite matrix $\mathbf{B} = ((b_{ij}))$ of order $n\times n$ can be written as $\mathbf{B} = \mathbf{A}\mathbf{A}^{\mathrm{T}}$, for some $n \times n$ non-singular matrix $\mathbf{A} = ((a_{ij}))$. With this choice of $\mathbf{A}$, the quantity appearing on the left-hand side of (1) equals $\det(\mathbf{B})$. Also, $b_{ii} = \sum_{j=1}^{n} a_{ij}^2$ for every $i = 1, \ldots, n$. Thus, Theorem 1 above implies the following.

Theorem 2. *Let* $\mathbf{B} = ((b_{ij}))$ *be an* $n \times n$ *positive definite matrix. Then*

$$\det(\mathbf{B}) \leq \prod_{i=1}^{n} b_{ii}. \tag{2}$$

Remark 3. A close scrutiny of arguments leading to validity of Theorem 2 assuming Theorem 1 to hold (cf. Remark 2) will show that Theorem 2 implies Theorem 1. In other words, Theorems 1 and 2 are equivalent.

Remark 4. Since any non-negative definite matrix can be written as the limit of a sequence of $n \times n$ positive definite matrices, and also since both sides of (2) are continuous functions of the entries of $\mathbf{B}$, Theorem 2 is easily seen to hold *also* for any non-negative definite matrix.

Remark 5. Many proofs of Theorem 1 or of Theorem 2 are known to exist in literature. We describe two of them below, and mention briefly two more.

In perhaps the shortest proof of Theorem 2 (see Mirsky [3, p. 418]), one defines $\mathbf{C} = ((c_{ij}))$ by $c_{ij} = b_{ij}/\sqrt{b_{ii}b_{jj}}$. Then one notices that (2) is equivalent to the following: $\det(\mathbf{C}) \leq 1$. To see this, one notices that $\det(\mathbf{C}) = \alpha_1 \cdots \alpha_n$ and $1 = [(c_{11} + \cdots + c_{nn})/n]^n = [(\alpha_1 + \cdots + \alpha_n)/n]^n$, where $\alpha_1, \ldots, \alpha_n$ are the eigenvalues of $\mathbf{C}$. Since $\mathbf{C}$ is positive definite, each $\alpha_i > 0$. Hence, $[(\alpha_1 + \cdots + \alpha_n)/n]^n \geq \alpha_1 \cdots \alpha_n$, which, in turn, implies $\det(\mathbf{C}) \leq 1$.

In another proof of Theorem 2 (see Mirsky [3, pp. 416-417]), one defines $\mathbf{B}_k \stackrel{def}{=} ((b_{ij}))_{i,j=k,\ldots,n}$, $k = 1, \ldots, n$, and also $\mathbf{B}' \stackrel{def}{=} \mathbf{B}$ except the $(1,1)$-th entry of $\mathbf{B}'$ which is given by $b_{11} - \det(\mathbf{B})/\det(\mathbf{B}_2)$, and shows that $\mathbf{B}'$ is positive semi-definite. Consequently, $\det(\mathbf{B}) \leq b_{11}\det(\mathbf{B}_2)$, and the proof follows by induction.

Yet another proof of Theorem 2 (see Bellman [2, p. 126]) is as follows. Notice that $\det(\mathbf{B}) = b_{11}\mathbf{B}_{11} + \det(\mathbf{B}_{0,11})$, where $\mathbf{B}_{11}$ is the co-factor of b_{11} and $\mathbf{B}_{0,11}$ is the matrix obtained from $\mathbf{B}$ by making its $(1,1)$-th entry 0 and leaving all other entries unchanged. It is easy to verify, by noticing that the matrix obtained by deleting the first row and first column of $\mathbf{B}_{0,11}$ is positive definite, that $\det(\mathbf{B}_{0,11}) \leq 0$. The rest of the proof follows by induction.

The last proof we wish to mention (see Bellman [2, p. 127], Beckenbach and Bellman [1, pp. 63-64]) uses the relation between $\det(\mathbf{B})$ and the integral of $\exp(-\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x}/2)$ over $\mathbb{R}^n$, and some clever arguments.

We present in this paper a proof of Theorem 2 that uses in a straightforward manner spectral decomposition of a positive definite matrix and also the inequality between arithmetic mean and geometric mean of a set of positive numbers. The proof is provided in Section 2.

2. A Proof of Theorem 2

Suppose

$$\mathbf{B} = \mathbf{P}\mathbf{Ł}\mathbf{P}^{\mathrm{T}} \tag{3}$$

is a spectral decomposition of $\mathbf{B}$, where $\mathbf{P} = ((p_{ij}))$ is an $n \times n$ orthogonal matrix and Ł is an $n \times n$ diagonal matrix with Ł $= \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Each λ_i is positive.

Notice that

$$\det(\mathbf{B}) = \prod_{i=1}^{n} \lambda_i. \tag{4}$$

Also, (3) implies the following: $b_{ii} = \sum_{k=1}^{n} \lambda_k p_{ik}^2$ for $i = 1, \ldots, n$. Therefore,

$$\prod_{i=1}^{n} b_{ii} = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_n=1}^{n} \lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_n} p_{1k_1}^2 p_{2k_2}^2 \cdots p_{nk_n}^2. \tag{5}$$

Notice now that

$$\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_n=1}^{n} p_{1k_1}^2 p_{2k_2}^2 \cdots p_{nk_n}^2 = \prod_{j=1}^{n} \left(\sum_{k_j=1}^{n} p_{jk_j}^2 \right) = 1, \tag{6}$$

since $\sum_{k_1=1}^{n} p_{1k_1}^2 = \sum_{k_2=1}^{n} p_{2k_2}^2 = \cdots = \sum_{k_n=1}^{n} p_{nk_n}^2 = 1$, which is implied by orthogonality of $\mathbf{P}$. Moreover, $p_{1k_1}^2 p_{2k_2}^2 \cdots p_{nk_n}^2 \geq 0 \ \forall \ k_1, k_2, \ldots, k_n$. Hence,

$$\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_n=1}^{n} \lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_n} p_{1k_1}^2 p_{2k_2}^2 \cdots p_{nk_n}^2$$

is a weighted average of the following set of positive quantities:

$$\{\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_n} : k_j = 1, \ldots, n; j = 1, \ldots, n\}.$$

The weight associated with $\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_n}$ is $p_{1k_1}^2 p_{2k_2}^2\cdots p_{nk_n}^2$.

It follows from the preceding facts and the standard inequality between arithmetic mean and geometric mean of a set of positive numbers that

$$\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\cdots\sum_{k_n=1}^{n}\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_n}p_{1k_1}^2 p_{2k_2}^2\cdots p_{nk_n}^2 \geq \prod_{k_1=1}^{n}\prod_{k_2=1}^{n}\cdots\prod_{k_n=1}^{n}(\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_n})^{p_{1k_1}^2 p_{2k_2}^2\cdots p_{nk_n}^2}. \quad (7)$$

Letting $q_j \stackrel{def}{=} \prod_{k_1=1}^{n}\prod_{k_2=1}^{n}\cdots\prod_{k_n=1}^{n}\lambda_{k_j}^{p_{1k_1}^2 p_{2k_2}^2\cdots p_{nk_n}^2}$, $j=1,\ldots,n$, and $Q \stackrel{def}{=} \prod_{j=1}^{n} q_j$, we re-write (7) as

$$\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\cdots\sum_{k_n=1}^{n}\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_n}p_{1k_1}^2 p_{2k_2}^2\cdots p_{nk_n}^2 \geq Q. \quad (8)$$

Consider now the term $q_1 = \prod_{k_1=1}^{n}\prod_{k_2=1}^{n}\cdots\prod_{k_n=1}^{n}\lambda_{k_1}^{p_{1k_1}^2 p_{2k_2}^2\cdots p_{nk_n}^2}$ in Q. Notice that (cf. (6))

$$\sum_{k_2=1}^{n}\cdots\sum_{k_n=1}^{n}p_{1k_1}^2 p_{2k_2}^2\cdots p_{nk_n}^2 = \prod_{j=2}^{n}\left(\sum_{k_j=1}^{n}p_{jk_j}^2\right) = 1 \quad (9)$$

and so

$$\prod_{k_2=1}^{n}\cdots\prod_{k_n=1}^{n}\lambda_{k_1}^{p_{1k_1}^2 p_{2k_2}^2\cdots p_{nk_n}^2} = (\lambda_{k_1}^{p_{1k_1}^2})^{\sum_{k_2=1}^{n}\cdots\sum_{k_n=1}^{n}p_{2k_2}^2\cdots p_{nk_n}^2} = \lambda_{k_1}^{p_{1k_1}^2}. \quad (10)$$

It follows, therefore, from (9) and (10) that $q_1 = \prod_{k_1=1}^{n}\lambda_{k_1}^{p_{1k_1}^2}$. It can be proved in this manner that $q_j = \prod_{k_j=1}^{n}\lambda_{k_j}^{p_{jk_j}^2}$ for $j=1,\ldots,n$. This implies

$$Q = \prod_{i=1}^{n}\left(\prod_{k_i=1}^{n}\lambda_{k_i}^{p_{ik_i}^2}\right). \quad (11)$$

Notice now from (11) that $Q = \prod_{j=1}^{n}\left(\lambda_j^{\sum_{i=1}^{n}p_{ij}^2}\right)$. Moreover, orthogonality of $\mathbf{P}$ implies $\sum_{i=1}^{n}p_{ij}^2 = 1$ for every $j=1,\ldots,n$. Hence,

$$Q = \lambda_1\lambda_2\cdots\lambda_n. \quad (12)$$

The theorem follows now from (4), (5), (8), and (12).

Remark 6. The proof presented in this paper relies on spectral decomposition of a positive definite matrix and also on the standard inequality between arithmetic mean and geometric mean of a set of positive numbers, but does not depend on any trick. Even though our proof may look complicated at a first sight, we believe that on close inspection it will be found to be simple.

References

[1] E.F. Beckenbach, R. Bellman, *Inequalities*, Springer-Verlag, Berlin (1961).

[2] R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York (1960).

[3] L. Mirsky, *An Introduction to Linear Algebra*, Clarendon, Oxford (1955).