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ON THE PROXIMATE DEFICIENCY OF MEROMORPHIC FUNCTIONS

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Abstract: The proximate deficiency of meromorphic functions on the basis of sharing of values of them is studied in this paper.

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1. Introduction, Definitions and Notations

Let f and g be two non-constant meromorphic functions defined on the open complex plane $\mathbb{C}$. If for $a \in \mathbb{C} \cup \{\infty\}$ f and g have the same a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities).

The following definitions are well known.

Definition 1. We denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points for $a \in \mathbb{C} \cup \{\infty\}$.

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Definition 2. Let k be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid \geq k)$ the counting function of those a-points of f whose multiplicities are not less than k, where an a-point is counted according to its multiplicity. Also by $\bar{N}(r, a; f \mid \geq k)$ we denote the corresponding reduced counting function.

We denote by $N(r, a; f \mid \leq k)$ the counting function $N(r, a; f) - N(r, a; f \mid \geq k+1)$.

Definition 3. We put

$$\delta_{1)}(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f \mid = 1)}{T(r, f)}$$

and

$$\delta_{2)}(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f \mid \leq 2)}{T(r, f)}$$

for $a \in \mathbb{C} \cup \{\infty\}$.

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. The order ρ_f of f is defined as

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna's characteristic function of f. When $\rho_f < \infty$ then f is of finite order.

A function $\rho_f(r)$ is called a proximate order of f if the following conditions hold:

(i) $\rho_f(r)$ is non-negative and continuous for $r > r_0$, say.

(ii) $\rho_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\rho'_f(r+0)$ and $\rho'_f(r-0)$ exist,

(iii) $\lim_{r \to \infty} \rho_f(r) = \rho_f$,

(iv) $\lim_{r \to \infty} r\rho'_f(r) \log r = 0$ and

(v) $\limsup_{r \to \infty} \frac{T(r,f)}{r^{\rho_f(r)}} = 1$.

The existence of such a proximate order is proved by Lahiri [3].

We now introduce the following definition.

Definition 4. The quantities $\delta_{1)\rho_f}(a; f)$ and $\delta_{2)\rho_f}(a; f)$ are defined as follows

$$\delta_{1)\rho_f}(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f \mid = 1)}{r^{\rho_f(r)}}$$

and

$$\delta_{2)\rho_f}(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f \mid\leq 2)}{r^{\rho_f(r)}}$$

for $a \in \mathbb{C} \cup \{\infty\}$.

In the paper we intend to establish a few results on the proximate deficiency of meromorphic functions involving sharing of values of them. We do not explain the definitions and standard notations of Nevanlinna theory because those are available in Hayman [1]. The term $S(r,f)$ denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. (see Yi [4]) *If f, g share $0, 1, \infty$ CM and $f \not\equiv g$ then:*

(i) $N(r,1;f \mid\geq 2) = N(r,1;g \mid\geq 2) = S(r,f) = S(r,g)$,

(ii) $N(r,0;f \mid\geq 2) = N(r,0;g \mid\geq 2) = S(r,f) = S(r,g)$, and

(iii) $N(r,\infty;f \mid\geq 2) = N(r,\infty;g \mid\geq 2) = S(r,f) = S(r,g)$.

Lemma 2. (see Hua and Fang [2]) *If f, g share $0, 1, \infty$ CM and $f \not\equiv g$ then for any complex number $a\,(\neq 0, 1, \infty)$:*

(i) $N(r,a;f \mid\geq 3) = S(r,f) = S(r,g)$ and

(ii) $N(r,a;g \mid\geq 3) = S(r,f) = S(r,g)$.

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *If f, g share $0, 1, \infty$ CM and $f \not\equiv g$ then:*

(i) $\delta_{1)\rho_f}(0;f) + \delta_{1)\rho_f}(1;f) + \delta_{1)\rho_f}(\infty;f) + \sum\limits_{a\neq 0,1,\infty} \delta_{2)\rho_f}(a;f) \leq 2$, and

(ii) $\delta_{1)\rho_g}(0;g) + \delta_{1)\rho_g}(1;g) + \delta_{1)\rho_g}(\infty;g) + \sum\limits_{a\neq 0,1,\infty} \delta_{2)\rho_g}(a;g) \leq 2$.

Proof. By Nevanlinna's Second Fundamental Theorem, Lemma 1 and Lemma 2 we get for pairwise distinct complex numbers $a_1, a_2, \cdots, a_n$ $(a_i \neq 0, 1, \infty;\ i = 1, 2, \cdots, n)$

$$(n+1)\,T(r,f) \le N(r,0;f\mid=1) + N(r,1;f\mid=1) + N(r,\infty;f\mid=1) + \sum_{i=1}^{n} N(r,a_i;f\mid\le 2) + S(r,f). \quad (1)$$

On dividing both sides of (1) by $r^{\rho_f(r)}$ and taking limit superior we obtain that

$$(n+1) \le \left\{1-\delta_{1)\rho_f}(0;f)\right\} + \left\{1-\delta_{1)\rho_f}(1;f)\right\} + \left\{1-\delta_{1)\rho_f}(\infty;f)\right\} + n - \sum_{a\ne 0,1,\infty} \delta_{2)\rho_f}(a;f),$$

from which (i) follows.

In a similar manner we can prove (ii). □

Theorem 2. *If f,g share $0,1,\infty$ CM and $f \not\equiv g$ then:*

(i) $\delta_{1)}(0;f) + \delta_{1)}(1;f) + \delta_{1)}(\infty;f) + \sum_{a\ne 0,1,\infty} \delta_{2)}(a;f) \le 2$, *and*

(ii) $\delta_{1)}(0;g) + \delta_{1)}(1;g) + \delta_{1)}(\infty;g) + \sum_{a\ne 0,1,\infty} \delta_{2)}(a;g) \le 2$.

Proof. Dividing both sides of (1) by $T(r,f)$ and taking limit superior we get that

$$(n+1) \le \left\{1-\delta_{1)}(0;f)\right\} + \left\{1-\delta_{1)}(1;f)\right\} + \left\{1-\delta_{1)}(\infty;f)\right\} + n - \sum_{a\ne 0,1,\infty} \delta_{2)}(a;f),$$

from which (i) follows.

Similarly we can prove (ii). □

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