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THE BRILL-NOETHER THEORY OF CURVES OF QUASI-COMPACT TYPE

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Abstract: Let X be a reduced nodal projective curve whose dual graph is either a tree or a cycle. Here we study the Brill-Noether theory for certain rank 1 torsion free sheaves on X (roughly speaking, we assume that they are spanned at all singular points of X).

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1. Introduction

Let X be a connected projective curve with arithmetic genus g. Let $\mathrm{Sing}(X)''$ denote the set of all singular points lying in at least two irreducible components of X. The curve X is said to be *quasi-compact* if there is an ordering $T_1, \dots, T_s$ of the irreducible components such that for all integers $i \in \{2, \dots, s\}$ the curve $Y_{i-1} := T_1 \cup \dots \cup T_{i-1}$ is connected, $\sharp(T_i \cap Y_{i-1}) = 1$ and the point $T_i \cap Y_{i-1}$ is an ordinary node of X. Hence every integral curve is quasi-compact. If X is quasi-compact, then each point of $\mathrm{Sing}(X)''$ is an ordinary node of X. The curve X is said to be a *loop* if each point of $\mathrm{Sing}(X)''$ is an ordinary node of X and there is an ordering $T_1, \dots, T_s$ of the irreducible components of X such that either $s = 2$ and $\sharp(T_1 \cap T_2) = 2$ or $s \geq 3$, $\sharp(T_i \cap T_j) \leq 1$ for all $i \neq j$ and $T_i \cap T_j \neq \emptyset$ if and only if either $|i - j| \leq 1$ or $\{i, j\} = \{1, s\}$.

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For the elementary properties of depth 1 coherent sheaves on reduced curves, see [3], parts VII and VIII. We say that a depth 1 sheaf F on X has pure pure rank 1 if its restriction to X_{reg} is a pure rank 1 vector bundle. Form now on we assume that each point of $\mathrm{Sing}(X)''$ is an ordinary node of X. Let F be a coherent sheaf on X with depth 1 and pure rank 1. The degree $\deg(F)$ of F satisfies the Riemann-Roch formula $\chi(F) = \deg(F) + \chi(\mathcal{O}_X)$. We will often call $\deg(F)$ the total degree of F. Set $\mathrm{Sing}(F) := \{P \in X : F$ is not locally free at $P\}$. Notice that $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$. Set $\mathrm{Sing}(F)'' := \mathrm{Sing}(F) \cap \mathrm{Sing}(X)''$. Fix $P \in \mathrm{Sing}(X)''$. The classification of rank 1 depth 1 modules on a ordinary node (see [3], pp. 164–166) gives that the germ F_P of F at P is isomorphic to the maximal ideal of the local ring $\mathcal{O}_{X,P}$. For any $S \subseteq \mathrm{Sing}(X)''$ let $u_S : C_S \to X$ denote the partial normalization of X in which we normalize only the points of S. Quite often C_S is disconnected. We always have $\chi(\mathcal{O}_{C_S}) = \chi(\mathcal{O}_X) + \sharp(S)$. If $S = \mathrm{Sing}(F)''$, then we often write u_F and C_F instead of u_S and C_S. Set $\widetilde{F} := u_F^*(F)/\mathrm{Tors}(u_F^*(F))$. Notice that $\widetilde{F}$ is a coherent sheaf on C_F with depth 1 and pure rank 1 and that $\mathrm{Sing}(\widetilde{F})'') = \emptyset$. We have $\deg(\widetilde{F}) = \deg(F) - \sharp(\mathrm{Sing}(F)'')$. Now assume that X has s irreducible components and fix an ordering $T_1, \dots, T_s$ of them. We will call any element $\underline{d} = (d_1, \dots, d_s) \in \mathbb{Z}^{\oplus s}$ a multidegree or a multidegree for the line bundles of S. For any $L \in \mathrm{Pic}(X)$ we have $\deg(L) = \sum_{i=1}^{s} \deg(L|T_i)$. This formula also holds for any coherent sheaf on X with depth 1 and pure rank 1 such that $\mathrm{Sing}(L)'' = \emptyset$, but fails in general. We may apply the formula to $\widetilde{F}$ on the curve C_F. We will say that F has multidegree $(\underline{d}, \mathrm{Sing}(F)'')$, where $\underline{d}$ is the multidegree of $\widetilde{F}$ with respect to the ordering $Y_1, \dots, Y_s$ of the irreducible components of C_F such that $u_F(Y_i) = T_i$ for all i. We will identify $\underline{d}$ with $(\underline{d}, S)$.

Theorem 1. *Let X be a projective curve of quasi-compact type with respect to an ordering $T_1, \dots, T_s$, $s \ge 2$, of its irreducible components. Fix integers $a \ge 2$, $d_i \ge 0$, $1 \le i \le s$, and a sets $S \subseteq \mathrm{Sing}(X)''$. Set $\underline{d} := (d_1, \dots, d_s)$. Let $A^a_{(\underline{d},S)}(T_1, \dots, T_s)$ denote the set of all coherent sheaves F on X with depth 1, pure rank 1, $\mathrm{Sing}(F)'' = S$, multidegree $\underline{d}$, $h^0(X, F) = a$, and F is spanned at each point of $\mathrm{Sing}(X)''$. Then $A^a_{(\underline{d},S)}(T_1, \dots, T_s) = \emptyset$ if $a < \sharp(S)$, while if $a \ge \sharp(S)$, then $A^a_{(\underline{d},S)}(T_1, \dots, T_s)$ is the disjoint union over all $\underline{a} = (a_1, \dots, a_s) \in \mathbb{Z}^{\oplus s}$ such that $a_i > 0$ for all i and $\sum_{i=1}^{s} a_i = a - \sharp(S) + s$ of the following set $B^{\underline{a}}_{\underline{d}}(T_1, \dots, T_s)$, where $B^{\underline{a}}_{\underline{d}}(T_1, \dots, T_s) = \prod_{i=1}^{s} A^{a_i}_{d_i}(T_i, T_i \cap (\overline{X \backslash T_i}) \cap S)$ and $A^{a_i}_{d_i}(T_i, T_i \cap \overline{X \backslash T_i})$ is the set of all coherent sheaves F_i on T_i with depth 1, rank 1, $\deg(F_i) = d_i$, $h^0(T_i, F_i) = a_i$ and F_i is spanned by its global sections at each point of $T_i \cap \overline{X \backslash T_i} \cap S$.*

Theorem 2. *Let X be a loop with respect to an ordering $T_1,\dots,T_s$ of its irreducible components. Fix integers $a \ge 2$, $d_i \ge 0$, $1 \le i \le s$, and a sets $S \subseteq \mathrm{Sing}(X)''$. Set $\underline{d} := (d_1,\dots,d_s)$. Let $A^a_{(\underline{d},S)}(T_1,\dots,T_s)$ denote the set of all coherent sheaves F on X with depth 1, pure rank 1, $\mathrm{Sing}(F)'' = S$, multidegree $\underline{d}$, $h^0(X,F) = a$ and F is spanned at each point of S. First assume $S \neq \emptyset$ and fix $P \in S$. Set $S' := u^{-1}_{\{P\}}(S\backslash\{P\})$ and let M_i be the unique irreducible component of $C_{\{P\}}$ such that $u_{\{P\}}(M_i) = T_i$. The curve $C_{\{P\}}$ is connected and quasi-compact with respect to the ordering $M_1,\dots,M_s$ of its irreducible components. There is a bijection (given by the map $F \to u^*_{\{P\}}(F)/\mathrm{Tors}(u^*_{\{P\}}(F))$ and its inverse $G \to u_{\{P\}*}(G)$) between the set $A^a_{(\underline{d},S)}(T_1,\dots,T_s)$ and the set $A^a_{(\underline{d},S')}(M_1,\dots,M_s)$. Now assume $S = \emptyset$. Set $\{P_i,Q_i\} := T_i \cap \mathrm{Sing}(X)''$. Let L_i, $1 \le i \le s$, be a rank 1 torsion free sheaf on T_i which is locally free at P_i and Q_i. Let $S(L_1,\dots,L_s)$ denote the set of all isomorphic classes of coherent sheavesL on X with depth 1, pure rank 1, locally free at each point of $\mathrm{Sing}(X)''$ such that $L|T_i \cong L_i$. Since X is a loop, the set $S(L_1,\dots,L_s)$ is an algebraic variety isomorphic to $\mathbb{C}_m$.*

(i) Let L be any line bundles on X which is spanned at each point of $\mathrm{Sing}(X)''$. Then

$$\sum_{i=1}^{s} h^0(T_i, L|T_i) - s \le h^0(X,L) \le \sum_{i=1}^{s} h^0(T_i, L|T_i) - s + 1\,. \tag{1}$$

If $h^0(X,L) = \sum_{i=1}^s h^0(T_i,L|T_i) - s$ (resp. $h^0(X,L) = \sum_{i=1}^s h^0(T_i,L|T_i) - s + 1$), then L is said to be of type I (resp. type II). L has type I (resp. type II) if and only if $h^1(X,L) = \sum_{i=1}^s h^1(T_i,L|T_i) - 1$ (resp. $h^1(X,L) = \sum_{i=1}^s h^0(T_i,L|T_i)$.

(ii) All type I elements of $A^a_{(\underline{d},\emptyset)}(T_1,\dots,T_s)$ are described in the following way. Let L_i, $1 \le i \le s$, be a rank 1 torsion free sheaf on T_i which is locally free at P_i and Q_i and $\deg(L_i) = d_i$. Assume the existence of $j \in \{1,\dots,s\}$ such that $h^0(T_j, \mathcal{I}_{\{P_j,Q_j\}} \otimes L_j) = h^0(T_j,L_j) - 2$. Then every element of $S(L_1,\dots,L_s)$ is a type I element of $A^a_{(\underline{d},S)}(T_1,\dots,T_s)$. Conversely, fix L of type I. Then $L_i := L|T_i$ are spanned at $\{P_i,Q_i\}$ and there is $j \in \{1,\dots,s\}$ such that $h^0(T_j,\mathcal{I}_{\{P_j,Q_j\}} \otimes L_j) = h^0(T_j,L_j) - 2$.

(iii) All locally free type II elements of $A^a_{(\underline{d},\emptyset)}(T_1,\dots,T_s)$ which are spanned at each singular point of X are described in the following way. Fix non-negative Cartier divisors $D_i \subset (T_i)_{reg}\backslash\{P_i,Q_i\}$ and spanned $M_i \in \mathrm{Pic}(T_i)$, $1 \le i \le s$, such that $\deg(M_i) = d_i - \deg(D_i)$ and $h^0(T_i,\mathcal{I}_{P_i} \otimes L_i) = h^0(T_i, \mathcal{I}_{\{P_i,Q_i\}} \otimes L_i)$ for all i. There is a unique $M \in S(M_1,\dots,M_s)$ such that $L := M(D)$ a type II element of $A^a_{(\underline{d},\emptyset)}(T_1,\dots,T_s)$, where $D := \sum_{i=1}^s D_i$; morover, D is its base-scheme. Conversely, if L has type II, it is locally free and its base scheme D

is a Cartier divisor, then $L \in S(L_1(-D_1), \dots, L_s(-D_s))$*, where* $L_i := L|T_i$*,* $D_i := D \cap T_i$*,* $L(-D_i)$ *is spanned at* $\{P_i, Q_i\}$*,* $h^0(T_i, L_i(-D_i)) = h^0(T_i, L_i)$ *and* $h^0(T_i, \mathcal{I}_{P_i} \otimes L_i) = h^0(T_i, \mathcal{I}_{\{P_i,Q_i\}} \otimes L_i)$ *for all* i*.*

Notice that the assuption "which are spanned at each singular point of X" in part (iii) of the statement of Theorem 2 is always satisfied if $\mathrm{Sing}(X) = \mathrm{Sing}(X)''$, i.e. if each T_i is smooth.

Take the set-up of Theorems 1 or 2. Since $s \geq 2$ and X is connected, $T_i \cap \overline{X \backslash T_i} \neq \emptyset$ for all i. Since F is assumed to be spanned at each point of $\mathrm{Sing}(X)''$, we need to assume $a_i > 0$ for all i.

At the end of the paper we spell out the cases in which each T_i is an elliptic curve and X is either of quasi-compact type (and hence of compact type) (see Example 1) or X is a loop (see Example 2).

2. Proofs and Examples

Remark 1. Let X be a curve with x irreducible components $M_1, \dots, M_x$, $x \geq 2$, such that each point of $\mathrm{Sing}(X)''$ is an ordinary node of X. For each $i \in \{1, \dots, x\}$ fix a rank 1 torsion free sheaf G_i on M_i which is locally free at each point of $M_i \cap \mathrm{Sing}(X)''$. Since a node is a universal singularity in the sense of [1], p. 387, there is a coherent sheaf G on X with depth 1, pure rank 1, $\mathrm{Sing}(G)'' = \emptyset$ and $G|M_i \cong G_i$. If each connected component of X is of quasi-compact type, then G is unique, up to isomorphisms.

Proof of Theorem 1. Fix $F \in A^a_{(\underline{d},S)}(T_1, \dots, T_s)$. Hence $\mathrm{Sing}(F)'' = S$. Since the tensor product is a right exact functor, the sheaf $u_S^*(F)$ is spanned at each point of $u_S^{-1}(\mathrm{Sing}(X)'')$. Hence $\widetilde{F}$ is spanned at each point of $u_S^{-1}(\mathrm{Sing}(X)'')$ and in particular at all points of $\mathrm{Sing}(C_S)''$. Let Y_i be the only irreducible component of C_S such that $u_S(Y_i) = T_i$. Set $a_i := h^0(Y_i, \widetilde{F}|Y_i)$. For every integer i such that $1 \leq i \leq s$ set $U_i := Y_1 \cup \dots \cup Y_i$. Since $\widetilde{F}$ is locally free at each point of $\mathrm{Sing}(C_S)''$, for every integer $i \in \{2, \dots, s\}$ we have the Mayer-Vietoris exact sequences

$$0 \to \widetilde{F}|U_i \to \widetilde{F}|U_{i-1} \oplus \widetilde{F}|T_i \to \widetilde{F}|U_{i-1} \cap Y_i \to 0 \,. \tag{2}$$

First assume $U_{i-1} \cap Y_i \neq \emptyset$, i.e. $(T_1 \cup \dots \cup T_{i-1}) \cap T_i \notin S$. Since $\widetilde{F}$ is spanned at each point of $\mathrm{Sing}(C_S)''$, $\widetilde{F}|Y_i$ is spanned at each point of $U_{i-1} \cap Y_i$. Since $Y_{i-1} \cap T_i$ is a point, the restriction map $H^0(Y_i, \widetilde{F}|Y_i) \to H^0(Y_i, \widetilde{F}|U_{i-1} \cap Y_i)$ is surjective. Hence (2) gives $h^0(U_i, \widetilde{F}|U_i) = h^0(U_i, \widetilde{F}|U_{-1}) + h^0(U_i, \widetilde{F}|M_i) - 1$ and $h^1(U_i, \widetilde{F}|U_i) = h^1(U_i, \widetilde{F}|U_{-1}) + h^0(U_i, \widetilde{F}|M_i)$. If $U_{i-1} \cap Y_i = \emptyset$, then (2) gives

$h^0(U_i, \widetilde{F}|U_i) = h^0(U_i, \widetilde{F}|U_{-1}) + h^0(U_i, \widetilde{F}|M_i)$ and $h^1(U_i, \widetilde{F}|U_i) = h^1(U_i, \widetilde{F}|U_{-1}) + h^0(U_i, \widetilde{F}|M_i)$. At the end we get $h^0(C_S, \widetilde{F}) = \sum_{i=1}^s a_s + 1 - s + \sharp(S)$. □

Claim. $F \cong u_{S*}(\widetilde{F})$.

Proof of Claim. Since $\widetilde{F}$ is a quotient of $u_S^*(F)$ and u_{S*} is right-adjoint to u_S^* (see [2], Example II.1.18 and p. 110), there is a natural map $j : F \to u_{S*}(\widetilde{F})$. Since $\widetilde{F}$ is a quotient of $u_S^*(F)$ by a torsion sheaf and F has no torsion, j is injective. Obviuosly, j is an isomorphism at each point of $X \backslash S$. The surjectivity of j at each point of S follows from the classification of depth 1 modules of an ordinary nodal curve singularity (see [3], pp. 164–166) and the assumption $S \subseteq \mathrm{Sing}(F)$.

By Claim we get $h^0(X, F) = h^0(C_S, \widetilde{F})$ and hence we get that each element of $A^a_{(\underline{d},S)}(T_1, \ldots, T_s)$ induces a unique element of a unique $B^a_{\underline{d}}(T_1, \ldots, T_s)$. To get the opposite inclusion apply Remark 1 to C_S to get a suitable $\widetilde{F}$ and then set $F = u_{S*}(\widetilde{F})$. □

Proof of Theorem 2. Fix $P \in \mathrm{Sing}(X)''$. For each coherent sheaf F on X with depth 1 and pure depth 1 set $\widehat{F} := u^*_{\{P\}}(F)/\mathrm{Tors}(u^*_{\{P\}}(F))$. There is an inclusion $F \to u_{\{P\}*}(\widehat{F})$ which is bijective if and only if F is not locally free at P (use the classification of depth 1 modules of an ordinary nodal curve singularity (see [3], pp. 164–166)). This remark gives the case $S \neq \emptyset$ of the statement of Theorem 2

(a) From now on we assume $S = \emptyset$. Part (i) follows using $s-1$ Maier-Vietoris exact sequences (2). The only difference is in the last exact sequence, because $U_{s-1} \cap Y_s = U_{s-1} \cap T_s = \{P_s, Q_s\}$ (two points instead one point). Let L be a coherent sheaf on X with depth 1, pure rank 1, locally free and spanned at each point of $\mathrm{Sing}(X)''$. Obviously, each $L|T_i$ has the same properties. Fix rank 1 torsion free sheaves L_i, $1 \le i \le s$, locally free and spanned at P_i and Q_i and set $b := -s + \sum_{i=1}^s h^0(T_i, L_i)$. Fix any $L \in S(L_1, \ldots, L_s)$.

(b) Here we assume the existence of $j \in \{1, \ldots, s\}$ such that $h^0(T_j, \mathcal{I}_{\{P_j,Q_j\}} \otimes L_j) = h^0(T_j, L_j) - 2$, i.e. such that the restriction map

$$H^0(T_j, L_j) \to H^0(\{P_j, Q_j\}, L_j|\{P_j, Q_j\})$$

is surjective. Up to a cyclic permutation of the set $\{1, \ldots, s\}$ we reduce to the case $j = s$. As in step (a) the Mayer-Vietoris exact sequence (2) for $i = s$ gives $h^0(X, L) = b$. Inductively, we also get using $s-1$ times the Mayer-Vietoris exact sequence (2) that the restriction map $\rho_1 : H^0(X, L) \to H^0(T_1, L_1)$ is surjective. Taking a cyclic permutation of the set $\{1, \ldots, s\}$ we get the surjectivity of every map $\rho_i : H^0(X, L) \to H^0(T_i, L_i)$, $1 \le i \le s$. Hence L is spanned at each point of $\mathrm{Sing}(X)''$. Obviously, L is locally free at each point of $\mathrm{Sing}(X)''$. Hence every element of $S(L_1,\dots,L_s)$ is a type I element of $A^a_{(\underline{d},\emptyset)}(T_1,\dots,T_s)$ with $a:=b$.

(c) Now we show that the converse in part (ii) holds. We only need to check that if L has type 1, then there is $j\in\{1,\dots,s\}$ such that $h^0(T_j,\mathcal{I}_{\{P_j,Q_j\}}\otimes(L|T_j))=h^0(T_j,L|T_j)-2$. Assume that this is not true. The contradiction (and hence part (ii) of the statement of Theorem 2) and half of part (iii) of the statement of Theorem 2 follow at once from the following construction (hint: take $L_i:=L|T_i$ for all i). Here we assume $h^0(T_j,\mathcal{I}_{\{P_j,Q_j\}}\otimes L_j)\neq h^0(T_j,L_j)-2$ for all j. Since L_j is spanned at P_j and Q_j, every section of L_j vanishing at one of the points P_j or Q_j vanishes at the other point, too. Since the natural map $\eta : H^0(X,L)\to\oplus_{i=1}^s H^0(T_i,L_i)$ is injective, we get that any section of L vanishing at one point of $\mathrm{Sing}(X)''$ vanishes at all points of $\mathrm{Sing}(X)''$. Fix any $P\in\mathrm{Sing}(X)''$. The injectivity of η gives $h^0(X,\mathcal{I}_P\otimes L)=h^0(X,\mathcal{I}_{\mathrm{Sing}(X)''}\otimes L)$. Hence L is spanned at each point of $\mathrm{Sing}(X)''$ if and only if it is spanned at one of the points of $\mathrm{Sing}(X)''$, i.e. if and only if $h^0(X,L)=h^0(X,\mathcal{I}_P\otimes L)+1$ for some (or for all) $P\in\mathrm{Sing}(X)''$. Use $s-1$ Mayer-Vietoris exact sequences (2) to get that (with the assumption of this step (c)) $h^0(X,\mathcal{I}_P\otimes L)=\sum_{i=1}^s(h^0(T_i,L_i-1)=b$ for any $P\in\mathrm{Sing}(X)''$ and every $L\in S(L_1,\dots,L_s)$. Thus if there is $L\in S(L_1,\dots,L_s)$ which is spanned at one or at all points of $\mathrm{Sing}(X)''$, then it is of type II. To conclude the proof of parts (ii) and (iii) we need to check the existence of a unique $L\in S(L_1,\dots,L_s)$ which is spanned at one point of $\mathrm{Sing}(X)''$. Since D is a Cartier divisor, L is uniquely determined by $L(-D)$ and D. Hence we are reduced to the case $D=\emptyset$, i.e. $L_i=M_i$ for all i. Since L_i is a spanned line bundle, it defines a morphism $h_{L_i}:T_i\to\mathbb{P}^{n_i}$, $n_i:=h^0(T_i,L_i)-1$. Since every section of L_i vanishing at P_i vanishes at Q_i, $h_{L_i}(P_i)=h_{L_i}(Q_i)$. Fix $P\in\mathbb{P}^b$ and s linear subspaces $V_i\subset\mathbb{P}^b$, $1\le i\le s$, such that $\dim(V_i)=n_i$, $P\in V_i$ for all i, and $\mathbb{P}^b$ is spanned by $\cup_{i=1}^s V_i$. Notice that that $b=\sum_{i=1}^s n_i$, i.e. the linear spaces $V_1,\dots,V_s$ are as skew as possible, with the restriction that they pass through P. Identify each $\mathbb{P}^{n_i}$ with V_i sending the point $h_{L_i}(P_i)$ to P. In this way we get a unique set-theoretic map $\phi : X\to\mathbb{P}^b$ such that $\phi|T_i=L_i$ for all i. Since each point of $\mathrm{Sing}(X)''$ is an ordinary node, ϕ is a morphism. Set $L:=\phi^*(\mathcal{O}_{\mathbb{P}^n}(1))$. By construction L is a spanned line bundle, $h^0(X,L)\ge b+1$ and $L|T_i\cong L_i$ for every i. We saw in part (a) that $h^0(X,L)\le b+1$. Hence $h^0(X,L)=b+1$. Since L is spanned and locally free, $A^{b+1}_{(\underline{d},\emptyset)}(T_1,\dots,T_s)$. The universal property of projective spaces and part (a) show that L is the unique spanned line bundle such that $h^0(X,L)=b+1$ and $L|T_i\cong L_i$ for all i. We just saw the uniqeness of a spanned type II solution. Hence to conclude the proof it is sufficient to show that if $D=\emptyset$, then any type II solution is spanned. By the definition of our Brill-Noether locus each solution must be spanned at each point of $\mathrm{Sing}(X)''$. The condition $h^0(X,L) = b+1$ and part (a) show that each restriction map $H^0(X,L) \to H^0(T_i,L_i)$ is surjective. Since each L_i is spanned, and $X = \cup_{i=1}^{s} T_i$, L is spanned. □

Example 1. Assume that X is of quasi-compact type with respect to an ordering $T_1,\dots,T_s$ of its irreducible components and that each T_i is a smooth elliptic curve. Hence $p_a(X) = s$. Set $K_j := \{i \in \{1,\dots,s\}$ such that $d_i = j\}$ and $k_j := \sharp(K_j)$. Fix any $S \subseteq \mathrm{Pic}(X)''$. Let Y_i be the irreducible component of C_S such that $u_S(Y_i) = T_i$. $A^a_{(\underline{d},S)}(T_1,\dots,T_s) \neq \emptyset$ if and only if $d_i \geq 0$ for all i and $a = \sum_{i=1}^{s} d_i + k_0 + \sharp(S)$. If $A^a_{(\underline{d},S)}(T_1,\dots,T_s) \neq \emptyset$, then it is irreducible and of dimension $s - k_0$. More precisely, if $A^a_{(\underline{d},S)}(T_1,\dots,T_s) \neq \emptyset$, then it is parametrized by the s-ple of line bundles $(M_1,\dots,M_s)$ such that $M_i \in \mathrm{Pic}(Y_i)$ and $\deg(M_i|Y_i) = d_i$ for all i, with the only restriction that if $d_i = 0$, then $M_i \cong \mathcal{O}_{Y_i}$ and that if $d_i = 1$, then $M_i \neq \mathcal{O}_{Y_i}(P)$ for any $P \in Y_i \cap \mathrm{Sing}(C_S)''$. Notice that u_S induces a bijection between $Y_i \cap \mathrm{Sing}(C_S)''$ and $T_i \cap (\mathrm{Sing}(X)\backslash S)$. Moreover, this parametrization of $A^a_{(\underline{d},S)}(T_1,\dots,T_s)$ is one-to-one: any element of $A^a_{(\underline{d},S)}(T_1,\dots,T_s)$ is uniquely determined by the associated s-ple $(M_1,\dots,M_s)$ by the last assertion of Remark 1.

Example 2. Assume that X is a loop with respect to an ordering $T_1,\dots,T_s$ of its irreducible components and that each T_i is a smooth elliptic curve. Hence $p_a(X) = s+1$. If $S \neq \emptyset$ we saw in the first part of the proof of Theorem 2 how to reduce the study of $A^a_{(\underline{d},S)}(T_1,\dots,T_s)$ to the descrption of associated sets for disjoint unions of curves of quasi-compact type to which we may apply Example 1. In the case $S = \emptyset$ we only point out that if there is $j \in \{1,\dots,s\}$ such that $d_j \geq 2$, then we are in the case completely analyzed in step (b) of the proof of Theorem 2; all these solution are of type I. If $d_j \leq 1$ for all j, then the situation is very simple. Set $A := \{j \in \{1,\dots,s\} : d_j = 1\}$. For every $j \in A$ fix $B_j \in T_j\backslash\{P_j,Q_j\}$. Set $D := \sum_{j\in A} B_j$ and $L := \mathcal{O}_X(D)$. The line bundle L is a type II solution and all type II solution are obtained in this way. Notice that $h^0(X,L) = 1$ and that D is the base-scheme of L.

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