International Journal of Pure and Applied Mathematics
Volume 50 No. 4 2009, 585-589

A CONDITION FOR THE EXISTENCE OF H-STABLE VECTOR BUNDLES ON REDUCIBLE CURVES

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Abstract: Let X be a nodal projective curve whose irreducible components are smooth. Here we give a necessary condition on a polarization H for the existence of H-stable vector bundles on X with fixed rank and multidegree. If the dual graph of X is a tree we look at vector bundles on X with rank r and $k < r$ sections whose evaluation at every $P \in X$ are linearly independent.

AMS Subject Classification: 14H60, 14H20
Key Words: torsion free sheaf, reducible curve, stable vector bundle

1. Introduction

Let $\mathbb{K}$ be an algebraically closed base field such that char($\mathbb{K}$) = 0. Let X be a connected and nodal projective curve whose irreducible components are smooth. Fix a polarization H on X (see [6], p. 153–155). We would like to give criteria assuring the existence of an H-stable vector bundle with prescribed restrictions to the irreducible components of X. If X is stable, E has rank 1 and H is proportional to the canonical polarization, then the answer is purely combinatorial (L. Caporaso's Basic Inequality (see [2], or see [5], Definition 1.1). In the rank $r > 1$ case the situation is more difficult. We first give the following necessary condition.

Definition 1. Let X be a reduced and connected, but reducible projective curve. Fix an ordering $X_1, \dots, X_s$, $s \geq 2$, of the irreducible components of

Received: December 20, 2008

X. Set $g_i := p_a(X_i)$. For any vector bundle E on X the s-ple of integers $(d_1, \dots, d_s) := (\deg(E|X_1), \dots, \deg(E|X_s))$ is called the multidegree of E. Fix an integer $r \geq 1$ and integers d_i, $1 \leq i \leq s$. For every $S \subseteq X$ set $X_S := \cup_{i=1}^s X_i \subseteq X$. For every proper subset S of $\{1, \dots, s\}$ there is a restriction map $\rho_S : E \to E|X_S$. Set $M_S := \mathrm{Ker}(\rho_S)$. We have $\chi(M_S) = \chi(E) - \chi(E|X_S) = \sum_{j \notin S} d_j + r(p_a(X_S) - p_a(X))$. Since S is a proper subset of $\{1, \dots, s\}$, M_S is a proper subsheaf of E with multirank $(u_1, \dots, u_s)$, where $u_i = 0$ for all $i \in S$ and $u_i = r$ for all $j \notin S$. Let $(v_1, \dots, v_s)$ denote the multidegree of E. The s-ple $(v_1, \dots, v_s)$ only depends from X and the $(s+1)$-ple of integers $(r, d_1, \dots, d_s)$, not the choice of E. Let H be a polarization of X, i.e. take rational numbers $a_i > 0$, $1 \leq i \leq s$, such that $\sum_{i=1}^s a_i = 1$ (see [6], p. 153). H is said to be $(r; d_1, \dots, d_s)$-admissible (resp. $(r; d_1, \dots, d_s)$-semiadmissible) if $\sum_{j \notin S} v_j + r(p_a(X_S) - p_a(X))/r(\sum_{j \notin S} a_j) < \sum_{i=1}^s d_i + r(1 - p_a(X))/r$ (resp. $\sum_{j \notin S} v_j + r(p_a(X_S) - p_a(X))/r(\sum_{j \notin S} a_j) \leq \sum_{i=1}^s d_i + r(1 - p_a(X))/r$) for every proper subset S of $\{1, \dots, s\}$.

Remark 1. Take the set-up of Definition 1. Fix an integer $r \geq 1$ and integers d_i, $1 \leq i \leq s$. Let E be a rank r vector bundle on X with multidegree $(d_1, \dots, d_s)$. Let H be a polarization of X, i.e. take rational numbers $a_i > 0$, $1 \leq i \leq s$, such that $\sum_{i=1}^s a_i = 1$. Assume that E is H-stable (resp. H-semistable). Since each M_S is a depth 1 proper subsheaf of E which must be used to test the H-stability (resp. H-semistability of E) H must be $(r; d_1, \dots, d_s)$-admissible (resp. $(r; d_1, \dots, d_s)$-semiadmissible).

See Proposition 1 to see which part of the H-stability condition does not depend from the combinatorial structure of X.

From now on in the introduction we assume that the dual graph $\|X\|$ of S is a tree, i.e. we assume the existence of an ordering $X_1, \dots, X_s$ of the irreducible components of X such that each X_i is smooth and $X_i \cap (X_1 \cup \cdots \cup X_{i-1})$ is a unique point P_i for all $i \in \{2, \dots, s\}$. Set $g_i := p_a(X_i)$ and $g := p_a(X)$. Since $\|X\|$ is a tree, $g = g_1 + \cdots + g_s$. Fix integers r, d such that $r > 0$ and a smooth and connected projective curve C of genus $q \geq 2$. Let $M(C; r, d)$ denote the set of all stable vector bundles on C with rank r and degree d. The set $M(C; r, d)$ is a non-empty integral variety of dimension $r^2(q-1)+1$. Since $M(C; r, d)$ is a non-empty integral variety, we are allowed to consider its general element. For every vector bundle E on X its multidegree $\underline{\deg}(E)$ is the s-ple $(\deg(E|X_1), \dots, \deg(E|X_s)$. For every s-ple $(E_1, \dots, E_s)$ such that each E_i is a rank r vector bundle on X_i let $A(E_1, \dots, E_s)$ denote the set of all vector bundles E on X such that $E|X_i \cong E_i$ for all i. Since X is nodal and $\|X\|$ is a tree, $A(E_1, \dots, E_s)$ is a non-empty irreducible variety of dimension $(r-1)^2(s-1)$.

In Section 2 we prove the following result.

Theorem 1. *Assume $g_i \geq 2$ for all i. Fix integers $r > k > 0$ and d_i, $1 \leq i \leq s$, such that= $r \leq d_i + (r-k)g_i$ and $(r, d_i, k) \neq (r, r, r)$. Fix general $F_i \in M(X_i; r-k, d_i)$, $1 \leq i \leq s$, and a general $F \in A(F_1, \dots, F_s)$. Let E be a general extension of F by $\mathcal{O}_X^{\oplus k}$. Then $E|X_i \in M(X_i; r, d_i)$ for all i.*

2. The Proofs

Remark 2. Fix integers $r > r_1 > 0$, $e > 0$, and d and a smooth curve C of genus $q \geq 2$. Let E be a general element of $M(C; r, d)$. Let F be a rank r_1 saturated subsheaf of E. Set $x := \deg(F)$. By [3] or [4], Remark 3.14, we have $\mu(E/F) - \mu(F) \geq q-1$, i.e.

$$r_1 d - x(r - r_1) \geq (r - r_1) r_1 (q-1)\,. \tag{1}$$

Hence $\mu(F) \leq \mu(E) - e$ if $(r - r_1)(q-1) \geq er$. Hence even in the worst case $r_1 = r-1$ we have $\mu(F) \leq \mu(E) - e$ (resp. $\mu(F) < \mu(E) - e$) if $q \geq er+1$ (resp. $q \geq er+2$). We have $\chi(F)/r_1 = \mu(F) + q - 1$ and $\chi(E)/r = \mu(E) + q - 1$. Hence $\mu(F) \leq \mu(E) - e$ (resp. $\mu(F) < \mu(E) - e$) if and only if $\chi(F)/r \leq (\chi(E) - er)/r$ (resp. $\chi(F)/r_1 < (\chi(E) - er)/r$). Thus if $q \geq er+1$ (resp. $q \geq er+2$), then $\chi(F)/r_1 \leq (\chi(E) - e)/r$ (resp. $\chi(F)/r_1 < (\chi(E) - er)/r$) for any r_1.

Proposition 1. *Fix an integer $r > 0$. Let X be a connected and nodal projective curve whose irreducible components are smooth. Fix an ordering $X_1, \dots, X_s$ of the irreducible components of X and set $g_i := p_a(X)$, and $e_i := \sharp(X_i \cap \mathrm{Sing}(X))$. Assume $g_i \geq e_i r + 2$ (resp. $g_i \geq e_i r + 1$) for all i. Fix integers d_i, $1 \leq i \leq s$, any polarization H on X, and general $E_i \in M(X_i, r_i, d_i)$, $1 \leq i \leq s$. Let E be any vector bundle on X such that $E|X_i \cong E_i$ for all i. Then E is not H-destabilized (resp. H-semidestabilized) by any proper subsheaf F with multirank $(r_1, \dots, r_s)$ such that $0 < r_i < r$ for all i.*

Proof. For any $P \in \mathrm{Sing}(X)$ let $i(P), j(P) \in \{1, \dots, s\}$ denote the two indices such that $P \in X_{i(P)} \cap X_{j(P)}$ with the convention $i(P) < j(P)$. Fix a polarization H on X, i.e. fix $a_i \in \mathbb{Q}$, $1 \leq i \leq s$, such that $a_i > 0$ for all i and $\sum_{i=1}^{s} a_i = 1$. For any non-zero subsheaf A of E set $\mu_H(A) := \chi(A)/\sum_{i=1}^{s} a_i b_i$, where b_i is the rank of A at a general point of X_i (see [6], p. 153). The rational number $\mu_H(A)$ is called the H-slope of A. Assume that the thesis fails for some multirank $(r_1, \dots, r_s)$ such that $0 < r_i < r$ for all i and fix a proper subsheaf F of E with multirank $(r_1, \dots, r_s)$ and maximal H-slope. Hence $\mu_H(F) \geq \mu_H(E)$

(resp. $\mu_H(F) > \mu_H(E)$). By assumption we have

$$r \cdot \chi(F) \geq (\sum_{i=1}^{s} a_i r_i)\chi(E) \tag{2}$$

(resp. the strict inequality holds in (2)). For every $P \in \mathrm{Sing}(X)$ there is a unique integer a_P such that $0 \leq a_P \leq \min\{r_{i(P)}, r_{j(P)}\}$ and the germ F_P of F at P is isomorphic to the $\mathcal{O}_{X,P}$-module $\mathcal{O}_{X,P}^{\oplus a_P} \oplus \mathcal{O}_{X_{i(P)},P}^{\oplus(r_{i(P)}-a_P)} \oplus \mathcal{O}_{X_{j(P)},P}^{\oplus(r_{j(P)}-a_P)}$. Let $u : Y = X_1 \sqcup \cdots \sqcup X_s$ be the normalization map. For any coherent sheaf A on X with depth 1 set $A' := u^*(A)/\mathrm{Tors}(u^*(A))$. The coherent sheaf A' on Y has depth 1 and its generic ranks $(b_1, \dots, b_s)$ are the ones of A. We have $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) + \sharp(\mathrm{Sing}(X)) = \chi(\mathcal{O}_X) + (\sum_{i=1}^{s} e_i)/2$. Since E is locally free, we get $\chi(E') = \chi(E) + r(\sum_{i=1}^{s} e_i)/2$. Set $F_i := F'|X_i$, $1 \leq i \leq s$. Since $Y = \sqcup_{i=1}^{s} X_i$, we have $\chi(F') = \sum_{i=1}^{s} \chi(F_i)$. Recall that $0 < r_i < r$ for all i. Fix $i \in \{1, \dots, s\}$. Since $A_i \cong E_i$ and E_i is general in $M(X_i; r, d_i)$, we have $\chi(F_i) < \chi(A_i) - e_i r_i$ (resp. the weak inequality holds) if $g_i \geq e_i r + 2$ (resp. $g_i \geq e_i r + 1$) (Remark 2). We have $\sum_{i=1}^{s} \chi(F_i) \leq \chi(F) + \sum_{P \in \mathrm{Sing}(X)} a_P \leq \chi(F) + (\sum_{i=1}^{s} e_i r_i)/2$. Recall that $\chi(E) + r = \chi(E'') = \sum_{i=1}^{s} \chi(E_i)$. Put all these inequalities in (2) to get a contradiction. □

Remark 3. Take $X = X_1 \cup \cdots \cup X_s$ as in the statement of Theorem 1. For any $i \in \{1, \dots, \}$, set $X[i] := \cup_{1 \leq j \leq i} X_j$. Fix an integer $i \in \{2, \dots, s\}$ and any vector bundle G on $X[i]$. We have a Mayer-Vietoris exact sequence on $X[i]$:

$$0 \to G \to G|X[i-1] \oplus G|X_i \to G|\{P_i\} \to 0\,. \tag{3}$$

From (3) we get that if $h^1(X[i-1], G|X[i-1]) = h^1(X_i, G|X_i) = 0$ and $G|X_i$ is spanned at P_i, then $h^1(X[i-1], G) = 0$ and the restriction map $H^0(X[i], G) \to H^0(X[i-1], G|X[i-1]$ is surjective.

Proof of Theorem 1. Set $A := F \otimes \omega_X^{\oplus k}$ and $B_i := F_i \otimes \omega_{X_i}^{\oplus k}$. The effective divisor $\mathrm{Sing}(X) \cap X_i$ of X_i induces an inclusion $j_i : \omega_{X_i} \hookrightarrow \omega_X|X_i$ and hence an inclusion $u_i : B_i \hookrightarrow A|X_i$. Set $A := F \otimes \omega_X^{\oplus k}$. Let E_i be a general extension of F_i by $\mathcal{O}_{X_i}^{\oplus k}$. By [1], Theorem B, $E_i \in M(X_i, r, d)$. Hence it is sufficient to prove the existence of an extension E of F by $\mathcal{O}_X^{\oplus k}$ such that $E|X_i \cong E_i$ for all i. By the irreducibility of the vector space $\mathrm{Ext}^1(X; F, \mathcal{O}_X^{\oplus k})$ it is sufficient to prove that for every $c \in \{1, \dots, s\}$ the restriction map $\mathrm{Ext}^1(X; F, \mathcal{O}_X^{\oplus k}) \to \mathrm{Ext}^1(X_c; F_c, \mathcal{O}_{X_c}^{\oplus k})$ is surjective. By Serre duality and the existence of the inclusion u_c it is sufficient to prove the surjectivity of the restriction map $\rho_c : H^0(X, A) \to H^0(X_c, A|A_c)$. Since F_i is stable and of positive degree, $h^1(X_i, F_i \otimes \omega_{X_i}) = 0$ for all i. The generality of F_i implies $h^1(X_i, F_i \otimes \omega_{X_i}(-P_i)) = 0$ for all $i \geq 2$. Hence we may apply s times a Mayer-Vietoris exact sequence (3) and Remark 3. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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