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# BRILL-NOETHER THEORY OF RANK $R$ SHEAVES ON STABLE CURVES: AN EXTREMAL CASE

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**Abstract:** Let $X$ be a stable curve. Fix an integer $r > 0$. Here we give a bijection between the set of all disconnecting nodes of $X$ and the depth 1 sheaves with pure rank $r$ on $X$ satisfying cetain properties (among them $F$ spanned, $\deg(F) = r$, $h^0(X, F) \ge 2r$ and $F$ "maximally non-locally free"). Let $X_1$ and $X_2$ be the closures in $X$ of $X\backslash\}P\}$. If $F$ is $\omega_X$-semistable, then $p_a(X) = 2{\cdot}p_a(X_1) = 2{\cdot}p_a(X_2)$. The converse is true if $X_1$ and $X_2$ are ireducible.

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## 1. Introduction

Let $X$ be a projective curve with only ordinary nodes defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For the elementary properties of depth 1 coherent sheaves on reduced curves, see [5], parts VII and VIII. A depth 1 coherent sheaf $F$ on $X$ is said to have pure rank $r$ if $F|X_{reg}$ is a rank $r$ vector bundle. For any depth 1 sheaf on $X$ let $\mathrm{Sing}(F)$ denote the sert of all $P \in X$ such that $F$ is not locally free. Since $F$ has no torsion, $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$. Fix $P \in \mathrm{Sing}(X)$. Fix an integer $r > 0$ and an $\mathcal{O}_{X,P}$-module $M$ with depth 1 and pure rank $r$. Since $M$ has no torsion and pure rank $r$, there is an inclusion $j : M \hookrightarrow \mathcal{O}_{X,P}^{\oplus r}$ such that the $\mathcal{O}_{X,P}$-module $\mathcal{O}_{X,P}^{\oplus r}/j(M)$ has finite length, i.e. it is a finite-dimensional $\mathbb{K}$-vector space. Let $\ell_P(M)$ denote the minimal such dimension for all possible inclusions $j$. Hence $\ell_P(M) = 0$ if and only if $M$ is

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free. The classification of depth 1 sheaves on an ordinary node given in [5], pp. 163–166, (use the pure-rank case $r_1 = r_2$ in [5], Proposition VIII.3) gives that $M$ is uniquely determined, up to $\mathcal{O}_{X,P}$-isomorphisms, from the pair of integers $(r, \ell_P(M))$, that $\ell_P(M) \leq r$, and that $\ell_P(M) = r$ if and only if $M \cong m_{X,P}^{\oplus r}$, where $m_{X,P}$ denote the maximal ideal of the local ring $\mathcal{O}_{X,P}$. Here we prove the following results.

**Theorem 1.** *Fix integers $g \geq 2$ and $r \geq 1$. There are bijections $\Phi \xrightarrow{m} \Psi \xrightarrow{\nu} \Psi'$ between the following set $\Phi$, $\Psi$ and $\Psi'$:*

*(a) the set $\Phi$ of all pairs $(X, P)$, where $X \in \overline{\mathcal{M}}_g$ and $P$ is a disconnecting node of $X$, i.e. $P \in \mathrm{Sing}(X)$ such that $X_P$ has two connected components, where $u_P : X_P \to X$ is the partial normalization of $X$ in which we normalize only $P$;*

*(b) the set $\Psi$ of all pairs $(X, F)$, where $X \in \overline{\mathcal{M}}_g$, $F$ is a spanned depth 1 sheaf on $X$ with pure rank $r$, $h^0(X, F) = 2r$, $\sharp(\mathrm{Sing}(F)) = 1$, $\ell(F) = r$, and $\deg(F) = r$ (up to isomorphisms);*

*(c) the set $\Psi'$ of all pairs $(X, F)$, where $X \in \overline{\mathcal{M}}_g$, $F$ is a spanned depth 1 sheaf on $X$ with pure rank $r$, $h^0(X, F) \geq 2r$, $\sharp(\mathrm{Sing}(F)) = 1$, $\ell(F) = r$, and $\deg(F) = r$ (up to isomorphisms);*

*Moreover, for any $(X, F) \in \Psi'$ we have $F \cong F_1^{\oplus r}$, where $F_1$ is a degree 1 spanned sheaf on $X$ with depth 1 and rank 1 such that $h^0(X, F_1) = 2$. The map $\nu : \Psi :\to \Psi'$ is the natural inclusion. Fix any $(X, P) \in \Phi$. Then $m((X, P)) := (X, u_{P*}(\mathcal{O}_{X_P})^{\oplus r})$.*

**Proposition 1.** *In the set-up of Theorem 1 take $(X, F) = m((X, P))$ for some $(X, P) \in \Phi$. $F$ is $\omega_X$-semistable if $X$ has two irreducible components, say $X = X_1 \cup X_2$, and $p_a(X_1) = p_a(X_2) = p_a(X)/2$. If $F$ is $\omega_X$-semistable, then $g := p_a(X)$ is even and the closures $X_1$ and $X_2$ of the connected components of $X \backslash \{P\}$ in $X$ saisfies $p_a(X_1) = p_a(X_2) = g/2$.*

The case $r = 1$ is true (see [1], Theorem 2). Here we adapt the proof of the rank 1 case.

**Remark 1.** Let $T$ be an integral projective curve and $G$ a rank $r$ spanned vector bundle on $T$. Obviously, $\deg(G) \geq 0$. Now assume $\deg(G) = 0$. Since $G$ is spanned, $h^0(T, G) \geq \mathrm{rank}(G) = r$. Take a general $r$-dimensional linear subspace $V \subset H^0(T, G)$. Let $e_V : \mathcal{O}_Y \otimes V \to G$ be the evaluation map. Since $V$ is general and $G$ is spanned, $e_V$ is an isomorphism outside finitely many points of $T$. Hence $\deg(G) = \deg(\mathcal{O}_Y \otimes V) + \mathrm{length}(\mathrm{Coker}(e_V))(= \mathrm{length}(\mathrm{Coker}(e_V)))$. Since $\deg(G) \leq 0$, we get that $e_V$ is an isomorphism.

**Lemma 1.** *Let $Y$ be a reduced projective curve and $G$ a rank $r$ spanned vector bundle on $Y$. If $\deg(G) \le 0$, then $\deg(G) = 0$ and $G \cong \mathcal{O}_Y^{\oplus r}$.*

*Proof.* If $Y$ is irreducible, then the lemma is the last assertion of Remark 1. Hence we may assume that $Y$ is reducible. Let $\mathcal{B}(Y)$ denote the set of all irreducible components of $Y$. Since $G$ is locally free,

$$\deg(G) = \sum_{T \in \mathcal{B}(Y)} \deg(G|T)\,. \qquad (1)$$

Since $G$ is spanned, each $G|T$, $T \in \mathcal{B}(Y)$, is spanned. Hence $\deg(G|T) \ge 0$ for all $T$. Since $\deg(G) \le 0$, (1) implies $\deg(G) = 0$, $\deg(G|T) = 0$ for all $T$ and $G|T \cong \mathcal{O}_T^{\oplus r}$ for all $T$. Since $G$ is spanned, $h^0(X,G) \ge r$. Take a general $r$-dimensional linear subspace $V \subset H^0(Y,G)$. Let $e_V : \mathcal{O}_Y \otimes V \to G$ be the evaluation map. If $e_V$ drops rank at $Q \in Y$, then $e_V$ drops rank at each $T \in \mathcal{B}(Y)$ such that $Q \in T$, because $G|T$ is trivial. Since $V$ is general, it spans $Y$ outside finitely many points. Hence $e_V$ never drops rank. Since $e_V$ is a map between vector bundles with the same rank, we get that $e_V$ is an isomorphism. □

*Proof of Theorem 1.* Obviously, $\nu$ is injective. Fix $(X,P) \in \Phi$. Set $R := u_{P*}(\mathcal{O}_{X_P})$ and $m((X,P)) := R^{\oplus r}$. To check that $m((X,P)) \in \Psi$ it is sufficient to show that $R$ is spanned, $\deg(R) = 1$, $h^0(X,R) = 2$, and $P$ is the unique point of $X$ at which $R$ is not locally free. We have $h^0(X,R) = h^0(X_P,\mathcal{O}_{X_P}) = 2$, because $P$ is a disconnecting node. Since $u_P$ is finite, $R^1u_{P*}(\mathcal{O}_{X_P}) = 0$. Hence $\chi(\mathcal{O}_{X_P}) = \chi(R)$. Thus Riemann-Roch gives $\deg(R) = \chi(\mathcal{O}_{X_P}) - \chi(\mathcal{O}_X) = 1$. Since $\mathcal{O}_{X_P}$ is spanned, $R$ is spanned outside $P$. Since $X$ is connected and $h^0(X,R) > 1$, the subsheaf of $R$ spanned by $H^0(X,R)$ has degree $> 0$. Hence $R$ is spanned, concluding the construction of the map $m$. By construction $m$ is injective. To check that $\nu$ is surjective and $m$ is invertible, it is sufficient to define a map $\gamma : \Psi' \to \Phi$ such that $m \circ \gamma = \mathrm{Id}_{\Psi'}$. Fix $(X,F) \in \Psi'$. Call $P$ the unique point of $X$ at which $F$ is not locally free. Obviously, $P \in \mathrm{Sing}(X)$. Let $u_P : X_P \to X$ denote the partial normalization of $X$ in which we only normalize the point $P$. Set $G := u_P^*(F)/\mathrm{Tors}(u_P^*(F))$. Obviously, $G$ is locally free outside the two points $u_P^{-1}(P)$. Since these two points are smooth points of $X_P$ and $G$ has no torsion, $G$ is locally free also at these points. Hence $G$ is a rank $r$ vector bundle on $X$. It is easy to check that $\deg(G) = \deg(F) - \ell(F)$ (for an arbitrary $F$). Hence $\deg(G) = 0$. Since tensor product is a right exact functor and $F$ is spanned, $u_P^*(F)$ is spanned. Hence $G$ is spanned. Since $G$ is a rank $r$ vector bundle and $\deg(G) = 0$, $G \cong \mathcal{O}_{X_P}^{\oplus r}$ (Lemma 1). Hence $h^0(X_P,G) = ra$, where $a$ is the number of connected components of $X_P$. Since $X$ is nodal, either $a = 1$ or $a = 2$. Since $F$ has no torsion, the natural map $H^0(X,F) \to$

$H^0(X_P, G)$ is injective. Since $(X, F) \in \Psi'$, $h^0(X, F) \geq 2r$. Hence $a = 2$, i.e. $(X, P) := \gamma((X, F)) \in \Phi$. The first part of the proof gives $F \cong u_{P*}(G)$, i.e. $m \circ \gamma = \mathrm{Id}_{\Psi'}$. □

*Proof of Proposition 1.* Since $F \cong u_{P*}(\mathcal{O}_{X_P})^{\oplus r}$, $F$ is $\omega_X$ semistable if and only if $u_{P*}(\mathcal{O}_{X_P})$ is $\omega_X$ semistable. In the rank 1 case to check if $F_1 := u_{P*}(\mathcal{O}_{X_P})$ is $\omega_X$-semistable we may use Caporaso's Basic Inequality (see [4], Theorem 10.3.1). Let $X_1$ and $X_2$ be the closures in $X$ of the connected components of $X\backslash\{P\}$. Set $g := p_a(X)$ and $g_i := p_a(X_i)$. Since $P$ is a disconnecting node of $X$, $g = g_1 + g_2$, and $\deg(\omega_X|X_i) = 2g_i - 1$. By [3], Definition 1.1, if $F_1$ satisfies the Basic Inequality, then

$$(2g_i - 1)/(2g - 2) - 1/2 \leq 0 \leq (2g_i - 1)/(2g - 2) + 1/2 \qquad (2)$$

for $i = 1, 2$ and the converse holds if $X_1$ and $X_2$ are irreducible. Thus $g_1 = g_2 = g/2$ if and only if (2) are satisfied for all $i$. Hence $F_1$ is $\omega_X$-semistable if $X_1$ and $X_2$ are the only proper subcurves of $X$ for which we need to check the Basic Inequality. □

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## References

[1] E. Ballico, Low degree spanned sheaves with pure rank 1 on reducible curves, *Preprint*.

[2] L. Caporaso, A compactification of the universal Picard variety over the moduli space of stable curves, *J. Amer. Math. Soc.*, **7**, No. 3 (1994), 589-660.

[3] M. Melo, Compactified Picard stacks over $\overline{\mathcal{M}}_g$, *ArXiv:* math/0710.3008.

[4] R. Pandharipande, A compactification of the universal moduli space of slope-semistable vector bundles over $\overline{\mathcal{M}}_g$, *J. Amer. Math. Soc.*, **9**, No. 2 (1996), 425-471.

[5] C. Seshadri, Fibrés vectoriels sur les courbes algébriques, *Astérisque*, **96** (1982).