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MULTIPLE SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC SYSTEMS WITH NONSTANDARD GROWTH CONDITIONS

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Abstract: In this paper, we study the solutions of the $(p(x), q(x))$-Laplacian equations with Dirichlet boundary condition on a bounded domain, and obtain three solutions under appropriate hypotheses.

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1. Introduction and Main Result

In recent years, the study of differential equations and variational problems with $p(x)$-growth conditions has been an interesting topic, which arises from nonlinear electrorheological fluids (see [17]) and elastic mechanics (see [20]).

In this paper, we consider the quasilinear elliptic systems

$$\begin{cases} -\Delta_{p(x)}u = \lambda F_u(x,u,v) + \mu G_u(x,u,v) \ \text{ in } \ \Omega, \\ -\Delta_{q(x)}v = \lambda F_v(x,u,v) + \mu G_v(x,u,v) \ \text{ in } \ \Omega, \\ u = v = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{P}$$

where $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, $\lambda,\ \mu \in [0,\infty)$, $\Omega \subset R^N (N \geq 1)$ is a nonempty bounded open set with a boundary $\partial\Omega$

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of class C^1, $F, G : \Omega \times R \times R \to R$ are functions such that $F(\cdot, s, t), G(\cdot, s, t)$ are measurable in Ω for all $(s, t) \in R \times R$ and $F(x, \cdot, \cdot)$ is C^1 in $R \times R$ for a.e. $x \in \Omega$, F_u denotes the partial derivative of F with respect to u, G_u denotes the partial derivative of G with respect to u. And $p, q \in C(\overline{\Omega})$, $1 < p^- = \inf_{x \in \overline{\Omega}} p(x) \leq p^+ = \sup_{x \in \overline{\Omega}} p(x) < +\infty$, $1 < q^- = \inf_{x \in \overline{\Omega}} q(x) \leq q^+ = \sup_{x \in \overline{\Omega}} q(x) < +\infty$.

Moreover,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } \ p(x) < N\,, \\ \infty & \text{if } \ p(x) \geq N\,, \end{cases}$$

is the critical exponent just as in many papers. Obviously, $p(x) < p^*(x)$, $q(x) < q^*(x)$ for all $x \in \overline{\Omega}$.

In the sequel, E will denote the Cartesian product of two Sobolev spaces $W_0^{1,p(x)}(\Omega)$ and $W_0^{1,q(x)}(\Omega)$, i.e. $E = W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$. X will denote the Sobolev space $W_0^{1,p(x)}(\Omega)$.

In recent years, many publications [2]-[6], [11], [12], [18], [19] have appeared concerning quasilinear elliptic systems which have been used in a great variety of applications. Existence and multiplicity results for quasilinear elliptic systems with variational structure have been broadly investigated.

In [2], L. Boccardo and D. Figueiredo studied existence of critical points (minima and saddle points) of functionals of the type

$$\Phi(u, v) = \frac{1}{p} \int_\Omega |\nabla u|^p + \frac{1}{q} \int_\Omega |\nabla v|^q - \int_\Omega F(x, u, v)\,,$$

where p and q are real numbers larger than 1. That is to say the solutions of the system

$$\begin{cases} -\Delta_p u = F_u(x, u, v), \\ -\Delta_q v = F_v(x, u, v), \end{cases}$$

where p and q are real numbers larger than 1.

In [3], using the fibering method introduced by Pohozaev, Y. Bozhkova and E. Mitidieri proved the existence of multiple solutions for a Dirichlet problem associated with a quasilinear system involving a pair of (p, q)-Laplacian operators.

In [12], when $p(x) \equiv p$, $q(x) \equiv q$, $\mu = 0$ Chun Li and Chun-Lei Tang ensured the existence of three solutions for the problem

$$\begin{cases} -\Delta_p u = \lambda F_u(x, u, v) & \text{in } \Omega, \\ -\Delta_q v = \lambda F_v(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p > N$, $q > N$ and the F satisfies suitable assumptions.

The aim of this paper is to prove the following result:

Theorem 1. *Assume that there exist two positive constants c, d and two functions $\gamma(x), \beta(x) \in C(\overline{\Omega})$ with $1 < \gamma^- < \gamma^+ < p^-$, $1 < \beta^- < \beta^+ < q^-$ such that*

(j_1) *$F(x,s,t) \geq 0$ for a.e. $x \in \Omega$ and all $(s,t) \in [0,d] \times [0,d]$;*

(j_2) *$\exists$ $p_1(x), q_1(x) \in C(\overline{\Omega})$ and $p^+ < p_1^- \leq p_1(x) < p^*(x)$, $q^+ < q_1^- \leq q_1(x) < q^*(x)$ such that:*

$$\limsup_{(s,t)\to(0,0)} \sup_{x\in\Omega} \frac{F(x,s,t)}{|s|^{p_1(x)} + |t|^{q_1(x)}} < +\infty;$$

(j_3) *$|F(x,s,t)| \leq C(1+|s|^{\gamma(x)}+|t|^{\beta(x)})$ for a.e. $x \in \Omega$ and all $(s,t) \in R \times R$;*

(j_4) *$F(x,0,0) = 0$ for a.e. $x \in \Omega$. Then, there exist an open interval $\Lambda \subseteq [0,+\infty)$ and a positive real number r with the following property: for each $\lambda \in \Lambda$ and each function G: $\Omega \times R \times R \to R$, measurable in Ω, C^1 in $R \times R$ and satisfying*

$$\sup_{(x,s,t)\in\Omega\times R\times R} \frac{|G(x,s,t)|}{1 + |s|^{p_2(x)} + |t|^{q_2(x)}} < +\infty\,,$$

where $p_2, q_2 \in C(\overline{\Omega})$ and $p_2(x) < p^*(x), q_2(x) < q^*(x)$ for all $x \in \overline{\Omega}$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, problem (P) has at least three weak solutions whose norms in $W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ are less than r.

This paper is divided into three sections. In Section 2, we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces, and recall B. Ricceri's three critical points theorem. In Section 3, we give the proof of the main result.

2. Preliminaries

Let Ω be a bounded domain of $\mathbf{R}^N (N \geq 1)$ with a smooth boundary $\partial\Omega$. Set

$$C_+(\overline{\Omega}) = \{h \mid h \in C(\overline{\Omega}), \quad h(x) > 1 \ \text{ for all } \ x \in \overline{\Omega}\}.$$

$$L_+^\infty(\Omega) = \{p \in L^\infty(\Omega) : ess\inf_{x\in\Omega} p(x) > 1\}.$$

For $p \in L_+^\infty(\Omega)$, denote

$$p^- = p^-(\Omega) = \text{ess } \inf_{x\in\Omega} p(x), \quad p^+ = p^+(\Omega) = \text{ess } \sup_{x\in\Omega} p(x).$$

Define

$$L^{p(x)}(\Omega) = \left\{ u | u : \Omega \to \mathbf{R} \quad \text{is measurable and } \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty \right\}$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Define

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}\,.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

On the basic properties of the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ we refer to [7-10]. Here we display some facts which will be used later.

Proposition 2.1. (see [9]) (i) *The spaces* $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces*;

(ii) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(x)}$ to $L^{q(x)}(\Omega)$ *is compact and continuous*;

(iii) *There is a constant* C >0, *such that* $|u|_{p(x)} \le C|\nabla u|_{p(x)}$ $\forall$ $u \in W_0^{1,p(x)}(\Omega)$.

By (iii) of Proposition 2.1, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We will use $|\nabla u|_{p(x)}$ to replace $\|u\|$ in the following discussions.

Proposition 2.2. (see [7]) *Set* $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x$. *For* $u, u_k \in L^{p(x)}(\Omega)$, *we have:*

(1) For $u \ne 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;

(2) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$;

(3) If $|u|_{p(x)} > 1$, *then* $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$;

(4) If $|u|_{p(x)} < 1$, *then* $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;

(5) $\lim_{k\to\infty} |u_k|_{p(x)} = 0 \iff \lim_{k\to\infty} \rho(u_k) = 0$;

(6) $|u_k|_{p(x)} \to \infty \iff \rho(u_k) \to \infty$.

In this paper, the space E will be endowed with the following equivalent norm:

$$\|(u,v)\| = \|u\| + \|v\|,$$

where

$$\|u\| = \inf\left\{\lambda > 0 : \int_\Omega |\frac{\nabla u}{\lambda}|^{p(x)}dx \le 1\right\}, \|v\| = \inf\left\{\mu > 0 : \int_\Omega |\frac{\nabla v}{\mu}|^{q(x)}dx \le 1\right\}.$$

Similar to Proposition 2.2, we have:

Proposition 2.3. *Set* $\phi(u) = \int_\Omega |\nabla u|^{p(x)}dx$. *For* $u, u_k \in W^{1,p(x)}(\Omega)$, *we have:*

(1) For $u \neq 0$, $\|u\| = \lambda \Leftrightarrow \phi(\frac{u}{\lambda}) = 1$;

(2) $\|u\| < 1\ (= 1; > 1) \Leftrightarrow \phi(u) < 1\ (= 1; > 1)$;

(3) If $\|u\| > 1$, *then* $\|u\|^{p^-} \le \phi(u) \le \|u\|^{p^+}$;

(4) If $\|u\| < 1$, *then* $\|u\|^{p^+} \le \phi(u) \le \|u\|^{p^-}$;

(5) $\lim_{k\to\infty} \|u_k\| = 0 \iff \lim_{k\to\infty} \phi(u_k) = 0$;

(6) $\|u_k\| \to \infty \iff \phi(u_k) \to \infty$.

Let $G(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx$, $u \in X$. We denote $L = G' : X \to X^*$, then

$$(L(u), v) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx \quad \forall\ u, v \in X.$$

Proposition 2.4. (see [9]) (i) $L : X \to X^*$ *is a continuous, bounded and strictly monotone operator*;

(ii) L *is a mapping of type* (S_+), *i.e. if* $u_n \rightharpoonup u$ *in* X *and* $\overline{\lim}_{n\to\infty}(L(u_n) - L(u), u_n - u)) \le 0$, *then* $u_n \to u$ *in* X;

(iii) $L : X \to X^*$ *is a homeomorphism.*

Proposition 2.5. (see [16]) *Let* X *be a reflexive real Banach space;* $I \subseteq R$ *an interval;* $\Phi : X \to R$ *a sequentially weakly lower semi-continuous* C^1 *functional whose derivative admits a continuous inverse on* X^*; $J : X \to R$ *a* C^1 *functional with compact derivative. In addition,* Φ *is bounded on each bounded subset of* X. *Assume that*

$$\lim_{\|x\|\to+\infty} (\Phi(x) + \lambda J(x)) = +\infty \tag{1}$$

for all $\lambda \in I$, *and that there exists* $\rho \in R$ *such that*

$$\sup_{\lambda\in I}\inf_{x\in X}(\Phi(x) + \lambda(J(x) + \rho)) < \inf_{x\in X}\sup_{\lambda\in I}(\Phi(x) + \lambda(J(x) + \rho)). \tag{2}$$

Then, there exist a non-empty open set $A \subseteq I$ *and a positive real number* r *with the following property: for every* $\lambda \in A$ *and every* C^1 *functional* $\Psi : X \to R$

with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(x) + \lambda J'(x) + \mu \Psi'(x) = 0$$

has at least three solutions in X whose norms are less than r.

Proposition 2.6. (see [14]) *Let X be a non-empty set and Φ, J two real functionals on X. Assume that there are $\gamma > 0$, u_0, $u_1 \in X$, such that*

$$\Phi(u_0) = J(u_0) = 0, \qquad \Phi(u_1) > \gamma,$$

$$\sup_{u \in \Phi^{-1}((-\infty, \gamma])} J(u) < \gamma \frac{J(u_1)}{\Phi(u_1)}. \tag{3}$$

Then, for each ρ satisfying

$$\sup_{u \in \Phi^{-1}((-\infty, \gamma])} J(u) < \rho < \gamma \frac{J(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda(\rho - J(u)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda(\rho - J(u))).$$

3. Proof of the Main Result

Definition 3.1. We say that $(u, v) \in E$ is a weak solution of problem (P) if

$$\int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla \xi + |\nabla v|^{q(x)-2} \nabla v \nabla \eta) dx - \lambda \int_\Omega (F_u \xi + F_v \eta) dx - \mu \int_\Omega (G_u \xi + G_v \eta) dx = 0,$$

for $\forall$ $(\xi, \eta) \in E$. The corresponding energy functional of problem (P) is

$$H(u,v) = \Phi(u,v) + \lambda J(u,v) + \mu \Psi(u,v)$$
$$= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla v|^{q(x)} dx - \lambda \int_\Omega F(x,u,v) dx - \mu \int_\Omega G(x,u,v) dx,$$

where

$$\Phi(u,v) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla v|^{q(x)} dx;$$

$$J(u,v) = -\int_\Omega F(x,u,v) dx; \qquad \Psi(u,v) = -\int_\Omega G(x,u,v) dx.$$

Then, $H(u,v)$ is a C^1 functional and the critical points of it are weak solutions of the problem (P).

Proof of Theorem 1. Set Φ, J, Ψ as above. So, for each u, v, ξ, $\eta \in E$, one has

$$\Phi'(u,v)(\xi,\eta) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \xi dx + |\nabla v|^{q(x)-2}\nabla v \nabla \eta \mathrm{d}x\,,$$

$$J'(u,v)(\xi,\eta) = -\int_\Omega F_u(x,u,v)\xi \mathrm{d}x - \int_\Omega F_v(x,u,v)\eta \mathrm{d}x\,,$$

$$\Psi'(u.v)(\xi,\eta) = -\int_\Omega G_u(x,u,v)\xi \mathrm{d}x - \int_\Omega G_v(x,u,v)\eta \mathrm{d}x\,.$$

Hence, the weak solution of problem (P) are exactly the solutions of the equation

$$\Phi'(u,v) + \lambda J'(u,v) + \mu \Psi'(u.v) = 0.$$

From Proposition 2.4 (or [9] for details), of course, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on E^*, moreover, J and Ψ are continuously Gâteaux differentiable functional whose Gâteaux derivative are compact. Obviously, Φ is bounded on each bounded subset of X under our assumptions.

From Proposition 2.3, let $G(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx$ just as before, we have:

if $\|u\| \geq 1$, then

$$\frac{1}{p^+}\|u\|^{p^-} \leq G(u) \leq \frac{1}{p^-}\|u\|^{p^+}\,; \tag{4}$$

if $\|u\| < 1$, then

$$\frac{1}{p^+}\|u\|^{p^+} \leq G(u) \leq \frac{1}{p^-}\|u\|^{p^-}\,. \tag{5}$$

In fact, when $\|u\| < 1$ we can set $C_0 \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{1}{p^+}\|u\|^{p^+} \geq 0$, then we can get

$$G(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \geq \frac{1}{p^+}\|u\|^{p^-} - C_0\,.$$

It follows that

$$G(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \geq \frac{1}{p^+}\|u\|^{p^-} - C_0 \quad \forall\ u \in X\,.$$

So there exists a constant $C_1 \geq 0$, such that

$$\begin{aligned}\Phi(u,v) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{q(x)}|\nabla v|^{q(x)}dx \\ &\geq \frac{1}{p^+}\|u\|^{p^-} + \frac{1}{q^+}\|v\|^{q^-} - C_1\end{aligned}$$

holds for any $(u.v) \in E$.

$$\begin{aligned}\lambda J(u,v) &= -\lambda \int_\Omega F(x,u,v)dx \\ &\geq -\lambda \int_\Omega C(1 + |u|^{\gamma(x)} + |v|^{\beta(x)})dx \\ &\geq -\lambda C(|\Omega| + |u|^{\gamma^+}_{\gamma(x)} + |u|^{\gamma^-}_{\gamma(x)} + |v|^{\beta^+}_{\beta(x)} + |v|^{\beta^-}_{\beta(x)}) \\ &\geq -C_2(1 + |u|^{\gamma^+}_{\gamma(x)} + |v|^{\beta^+}_{\beta(x)}) \\ &\geq -C_3(1 + \|u\|^{\gamma^+} + \|v\|^{\beta^+})\end{aligned}$$

holds for any $(u.v) \in E$, where constants $C_2 \geq 0$, $C_3 \geq 0$. Here, we used the condition (j_3) and (ii) of Proposition 2.1. Combining the two inequalities above, we can get

$$\Phi(u,v) + \lambda J(u,v) \geq \frac{1}{p^+}\|u\|^{p^-} + \frac{1}{q^+}\|v\|^{q^-} - C_3(1 + \|u\|^{\gamma^+} + \|v\|^{\beta^+}) - C_1,$$

because of $\gamma^+ < p^-$, $\beta^+ < q^-$, it follows that

$$\lim_{\|(u,v)\| \to +\infty} (\Phi(u,v) + \lambda J(u,v)) = +\infty \quad \forall (u,v) \in E, \quad \lambda \in [0, +\infty)$$

Then the assumption (2.1) of Proposition 2.5 is satisfied.

Next, we will prove that the assumption (2.2) is also satisfied. It suffices to verify the conditions of Proposition 2.6. Let $(u_0, v_0) = (0,0)$, we can easily have

$$\Phi(u_0, v_0) = -J(u_0, v_0) = 0\,.$$

Now we claim that there exist $\gamma > 0$ and $(u_1, v_1) \in E$ such that $\Phi(u_1, v_1) > \gamma$ and (2.3) is satisfied.

There is a point $x^0 \in \Omega$ because of Ω is a nonempty bounded open set. Let $r_2 > r_1 > 0$, put

$$w(x) = \begin{cases} 0 & x \in \overline{\Omega} \setminus B(x^0, r_2)\,, \\ \frac{d}{r_2 - r_1}\left(r_2 - \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}\right) & x \in B(x^0, r_2) \setminus B(x^0, r_1)\,, \\ d & x \in B(x^0, r_1)\,. \end{cases}$$

Here, $B(x,r)$ stands for the open ball in R^N of radius r centered at x.

Let $(u_1(x), v_1(x)) = (w(x), w(x))$, then, thanks to (j_1) we can obtain that

$$-J(u_1, v_1) = -J(w, w) = \int_\Omega F(x, w, w) > 0.$$

From (j_2), $\exists\, \eta \in [0, 1]$, $C_1 > 0$, such that

$$\begin{aligned} F(x, s, t) &< C_1(|s|^{p_1(x)} + |t|^{q_1(x)}) \\ &< C_1(|s|^{p_1^-} + |t|^{q_1^-}) \quad \forall (s, t) \in [-\eta, \eta] \times [-\eta, \eta] \quad \text{a.e. } x \in \Omega. \end{aligned}$$

From (j_3), there are nine positive real number M_i $(i = 1, 2...9)$ according to $|s|, |t|$ larger or smaller than η and 1. For example, when $|s| > 1$, $|t| < \eta$ some

$$M_i = \sup_{|s|>1, |t|<\eta} \frac{C(1 + |s|^{\gamma^+} + |t|^{\beta^-})}{|s|^{p_1^-} + |t|^{q_1^-}}.$$

Let M=max$\{C_1, M_1......M_9\}$, then

$$F(x, s, t) < M(|s|^{p_1^-} + |t|^{q_1^-}) \quad \forall (s, t) \in R \times R \quad \text{a.e } x \in \Omega.$$

Consequently, fix γ such that $0 < \gamma < 1$. And when $\frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} \le \gamma < 1$, by the Sobolev Embedding Theorem ($X \hookrightarrow L^{p_1^-}(\Omega)$ is continuous), we have (for suitable positive constants C_2, C_3)

$$\begin{aligned} -J(u, v) = \int_\Omega F(x, u, v)dx &< M \int_\Omega (|u|^{p_1^-} + |v|^{q_1^-})dx \\ &\le C_2(\|u\|^{p_1^-} + \|v\|^{q_1^-}) \\ &\le C_3(\gamma^{\frac{p_1^-}{p^+}} + \gamma^{\frac{q_1^-}{q^+}}). \end{aligned}$$

Since $p_1^- > p^+, q_1^- > q^+$, we have

$$\lim_{\gamma \to 0^+} \frac{\sup_{\frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} \le \gamma} -J(u, v)}{\gamma} = 0. \tag{6}$$

We choose $w(x) \in X$ as above such that $-J(w, w) > 0$. Fix γ_0 such that $0 < \gamma < \gamma_0 < \min\{\frac{1}{p^+}, \frac{1}{q^+}\} \cdot \min\{\|w\|^{p^+} + \|w\|^{q^+},\ \|w\|^{p^-} + \|w\|^{q^-}, 1\} \le 1$. Then, we divide the proof into two cases.

(i) When $\|w\| < 1$, from (3.2) we have

$$\begin{aligned} \Phi(u_1, v_1) &= \Phi(w, w) \\ &= \int_\Omega \frac{1}{p(x)}|\nabla w|^{p(x)} + \frac{1}{q(x)}|\nabla w|^{q(x)}dx \\ &\ge \min\{\frac{1}{p^+}, \frac{1}{q^+}\} \cdot \int_\Omega |\nabla w|^{p(x)} + |\nabla w|^{q(x)}dx \end{aligned}$$

$$\begin{aligned} &\geq \min\{\frac{1}{p^+}, \frac{1}{q^+}\} \cdot (\|w\|^{p^+} + \|w\|^{q^+}) \\ &\geq \gamma_0 > \gamma. \end{aligned}$$

From(3.3), we know that

$$\begin{aligned} \sup_{\frac{1}{p^+}\|u\|^{p^+}+\frac{1}{q^+}\|v\|^{q^+}\leq\gamma} -J(u,v) &\leq \frac{\gamma}{2} \cdot \frac{-J(u_1,v_1)}{\max\{\frac{1}{p^-}, \frac{1}{q^-}\} \cdot (\|w\|^{p^-} + \|w\|^{q^-})} \\ &\leq \frac{\gamma}{2} \cdot \frac{-J(u_1,v_1)}{\Phi(u_1,v_1)} \\ &< \gamma \cdot \frac{-J(u_1,v_1)}{\Phi(u_1,v_1)}. \end{aligned}$$

(ii) When $\|w\| \geq 1$, then from (3.1) we have

$$\begin{aligned} \Phi(u_1,v_1) &= \Phi(w,w) \\ &= \int_\Omega \frac{1}{p(x)}|\nabla w|^{p(x)} + \frac{1}{q(x)}|\nabla w|^{q(x)} dx \\ &\geq \min\{\frac{1}{p^+}, \frac{1}{q^+}\} \cdot \int_\Omega |\nabla w|^{p(x)} + |\nabla w|^{q(x)} dx \\ &\geq \min\{\frac{1}{p^+}, \frac{1}{q^+}\} \cdot (\|w\|^{p^-} + \|w\|^{q^-}) \\ &\geq \gamma_0 > \gamma. \end{aligned}$$

From (3.3), we know that

$$\begin{aligned} \sup_{\frac{1}{p^+}\|u\|^{p^+}+\frac{1}{q^+}\|v\|^{q^+}\leq\gamma} -J(u,v) &\leq \frac{\gamma}{2} \cdot \frac{-J(u_1,v_1)}{\max\{\frac{1}{p^-}, \frac{1}{q^-}\} \cdot (\|w\|^{p^+} + \|w\|^{q^+})} \\ &\leq \frac{\gamma}{2} \cdot \frac{-J(u_1,v_1)}{\Phi(u_1,v_1)} \\ &< \gamma \cdot \frac{-J(u_1,v_1)}{\Phi(u_1,v_1)}. \end{aligned}$$

For any $(u,v) \in \Phi^{-1}((-\infty,\gamma])$, we can get $\Phi(u,v) \leq \gamma$, i.e.

$$\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{q(x)}|\nabla v|^{q(x)} dx \leq \gamma.$$

Then, we can get

$$\min\{\frac{1}{p^+}, \frac{1}{q^+}\} \cdot \int_\Omega |\nabla u|^{p(x)} + |\nabla v|^{q(x)} dx \leq \gamma.$$

So,

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)} + |\nabla v|^{q(x)} dx &< \gamma \cdot \frac{1}{\min\{\frac{1}{p^+}, \frac{1}{q^+}\}} \\ &< \gamma_0 \cdot \frac{1}{\min\{\frac{1}{p^+}, \frac{1}{q^+}\}} \\ &< 1.\end{aligned}$$

This inequality implies

$$\int_\Omega |\nabla u|^{p(x)} dx < 1 \qquad \int_\Omega |\nabla v|^{q(x)} dx < 1,$$

i.e.

$$\|u\| < 1 \qquad \|v\| < 1.$$

It follows that

$$\frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} < \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{q(x)}|\nabla v|^{q(x)} dx \le \gamma.$$

So we can get that

$$\Phi^{-1}((-\infty, \gamma]) \subset \left\{(u,v) : (u,v) \in E, \frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} < \gamma\right\}.$$

Then

$$\sup_{(u,v)\in\Phi^{-1}((-\infty,\gamma])} -J(u,v) \le \sup_{\frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} < \gamma} -J(u,v) < \gamma \cdot \frac{-J(u_1,v_1)}{\Phi(u_1,v_1)},$$

that is

$$\sup_{(u,v)\in\Phi^{-1}((-\infty,\gamma])} -J(u,v) < \gamma \cdot \frac{-J(u_1,v_1)}{\Phi(u_1,v_1)}.$$

So we can find $\gamma > 0$, $u_1 = v_1 = w$ and $\Phi(w,w) \le \gamma$ satisfying (2.3). Also we can find ρ satisfying

$$\sup_{(u,v)\in\Phi^{-1}((-\infty,\gamma])} -J(u,v) < \rho < \gamma \cdot \frac{-J(u_1,v_1)}{\Phi(u_1,v_1)}.$$

Set $I = [0, +\infty)$, moreover, $\Phi(u,v)$ and $-J(u,v)$ satisfy the assumption of Proposition 2.6, so using Proposition 2.6, we can easily obtain that (2.2) is satisfied.

Thus, Φ, J and Ψ satisfy all the assumptions of Proposition 2.5, and the proof is complete. □

Remark. Applying ([1], Theorem 2.1) in the proof of Theorem 1, an upper bound of the interval of parameters λ for which (P) has at least three weak solutions is obtained. To be precise, in the conclusion of Theorem 1 one has

$$\Lambda \subseteq \left[0, \frac{h\gamma}{\inf\limits_{(u,v)\in\Phi^{-1}((-\infty,\gamma])} J(u,v) - \gamma \cdot \frac{J(u_1,v_1)}{\Phi(u_1,v_1)}}\right]$$

for each $h > 1$ and (u_1, v_1) as in the proof of Theorem 1 (namely, $u_1 = v_1 = w$).

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