International Journal of Pure and Applied Mathematics
Volume 50 No. 4 2009, 609-613

ON WEAKLY COMPACT FRIENDLY OPERATORS

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Abstract: In this paper, it is introduced the definition of weakly compact-friendly operator and generalized some well-known results of compact-friendly, compact operators on Banach lattices.

AMS Subject Classification: 47A15, 47B60
Key Words: Banach lattice, AM-space, AL-space, weakly compact-friendly operator, invariant subspaces

1. Introduction

In this work, we investigate the results on weakly compact-friendly operators that is a generalization of compact-friendly operators.

Firstly, we introduce some known results, definitions and theorems.

Definition 1. An ordered vector space E is called a vector lattice if every pair of vectors has a supremum and an infimum. A (semi)norm p on a vector lattice E is said to be a lattice (semi)norm if $|x| \leq |y|$ implies $p(x) \leq p(y)$ for every $x, y \in E$. A vector lattice equipped with a complete lattice norm is referred to as a Banach lattice.

Definition 2. $T : E \to E$ is a bounded operator on a Banach space and V is a subspace of E. Then V is non-trivial if $V \neq \{0\}$ and $V \neq E$. V is invariant under T or simply T-invariant if $T(V) \subseteq V$. Also, V is said to be

Received: September 8, 3008
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T-hyperinvariant whenever V is invariant under every bounded operator on E that commutes with T.

Definition 3. A Banach lattice E is called an AM-space if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for every disjoint x and y in E. A Banach lattice E is called an AL-space if $\|x + y\| = \|x\| + \|y\|$ for every disjoint x and y in E.

Theorem 1. (see Aliprantis et al [4]) *If a positive operator B on a Banach lattice is dominated by a compact operator, then B^3 is compact.*

Theorem 2. (see Aliprantis et al [4]) *If a positive operator on a AM- or AL-space is weakly compact operator, then its square is compact.*

Theorem 3. (see Abramovich et al [2]) *Let $B, S : E \to E$ be two commuting non-zero positive operators on a Banach lattice. If one of them is quasinilpotent at a non-zero positive vector and the other dominates a non-zero compact operator, then B and S have common non-trivial closed invariant ideal.*

Definition 4. A positive operator $B : E \to E$ is said to be weakly compact-friendly if there exists a positive operator in the commutant of B dominates a non-zero operator which in turn is dominated by positive weakly compact operator. That is, B is weakly compact-friendly if and only if there exists three non-zero operators $R, C, K : E \to E$ with R, K positive and K weakly compact such that

$$RB = BR, \ \ |Cx| \leq R(|x|), \ \text{ and } \ |Cx| \leq K(|x|)$$

for every $x \in E$.

Lemma 1. (see Abramovich et al [2]) *For a Banach lattice with a quasi-interior $u > 0$ the following properties hold.*

(i) For every non-zero element $y \in E_u$ there exists an operator $V : E \to E$ which carries y to a non-zero positive vector and V is dominated by the identity operator, i.e.

$$V(y) > 0 \quad \text{and} \quad |Vx| \leq |x| \ \text{ for all } x \in E.$$

(ii) For every element v satisfying $0 \leq v \leq u$ there exists an operator $U : E \to E$ which carries u to v and U is dominated by identity operator, i.e.,

$$U(u) = v \ \text{ and } \ |Ux| \leq |x| \ \text{ for all } x \in E.$$

Of course, the zero subspace and the whole space are always invariant for every operator, so we will be looking for non-trivial invariant subspaces. It is easy to see that Ker T and Range T are T-hyperinvariant.

2. On Weakly Compact-Friendly Operators

Proposition. *Let $B : E \to E$ be a weakly compact-friendly operator and let $S : E \to E$ be an order isomorphism. Then the operator SBS^{-1} is also weakly compact-friendly.*

Proof. For an operator $T : E \to E$ and for an invertible operator $S : E \to E$ we will denote the composition STS^{-1} by T_S. A straightforward verification shows that if $R, T, K : E \to E$ are such that R and K are positive, K weakly compact, and

$$RB = BR, \ |Tx| \leq R(|x|), \ \text{and} \ |Tx| \leq K(|x|),$$

then for any order isomorphism $S : E \to E$ we have that R_S and K_S are positive, K_S is weakly compact, and

$$R_S B_S = B_S R_S \ , \ |T_S x| \leq R_S(|x|) \ , \ \text{and} \ |T_S x| \leq K_S(|x|).$$

Thus B_S is a weakly compact-friendly operator. □

Theorem 4. *If a non-zero weakly compact-friendly operator $B : E \to E$ on a AL- or AM-space E is quasinilpotent at some $x_0 > 0$, then B has a non-trivial closed invariant ideal.*

Moreover, if another positive operator $T : E \to E$ commutes with B, then T and B have a common non-trivial closed invariant ideal.

Proof. Let $B : E \to E$ be a positive non-zero weakly compact-friendly operator on a Banach lattice which is quasinilpotent at some $x_0 > 0$. Let $R, C, K : E \to E$ be non-zero operators with R, K positive and K weakly compact such that

$$RB = BR, \ |Cx| \leq R(|x|), \ \text{and} \ |Cx| \leq K(|x|)$$

for every $x \in E$.

Without loss of generality, we can suppose that $\|B + T\| < 1$ and define

$$A = \sum_{j=1}^{\infty} (B + T)^n.$$

Clearly, the positive operator A commutes with both B and T and satisfies $Ax \geq x$ for each $x \geq 0$. Also, for each $x > 0$, let $J[x]$ denote the non-zero principal ideal generated by Ax, i.e.

$$J[x] = \{y \in E : |y| \leq \lambda Ax \ \text{for some} \ \lambda > 0\}.$$

If $\overline{J[x]} \neq E$ for some $x > 0$, then ideal $\overline{J[x]}$ is a non-trivial closed $(B + T)$-invariant ideal. This ideal $\overline{J[x]}$ is, of course, also invariant under B and T. So, we can assume that

$$\overline{J[x]} = E$$

for each $x > 0$, i.e., Ax is a quasi-interior point in E for each $x > 0$.

Since $C \neq 0$ there exists some $x_1 > 0$ such that $Cx_1 \neq 0$. Since $A|Cx_1|$ is a quasi-interior point and $|Cx_1| \leq A|Cx_1|$ holds, it follows from Lemma 1, that there exists an operator $V_1 : E \to E$ dominated by the identity operator such that $x_2 = V_1 C x_1 > 0$. Put $M_1 = V_1 C$, and note that M_1 is dominated both by the weakly compact operator K and by the operator R.

From $\overline{J[x_2]} = E$ and $C \neq 0$, we see that there exists some element $0 \leq y \leq Ax_2$ such that $Cy \neq 0$. Since Ax_2 is a quasi-interior point, it follows from Lemma 1, that there exists an operator $U : E \to E$ dominated by the identity operator such that $UAx_2 = y$. Now note that $A|Cy|$ is a quasi-interior point. Since $|Cy| \leq A|Cy|$, it follows from Lemma 1 that there exists another operator $V_2 : E \to E$ dominated by the identity operator such that $x_3 = V_2 Cy = V_2 CUAx_2 > 0$. Let $M_2 = V_2 CUA$ and note that M_2 is dominated both by the weakly compact operator KA and by the operator RA.

If we repeat the preceding arguments with the vector x_2 replaced by x_3, then we obtain one more operator $M_3 : E \to E$ which satisfies $M_3 x_3 > 0$ and which is dominated both by the weakly compact operator KA and by the operator RA.

From $M_3 M_2 M_1 x_1 = M_3 x_3 > 0$, we see that $M_3 M_2 M_1$ is a non-zero operator. Since $|M_3 M_2 M_1(x)| \leq KAKAK(x)$ holds and KAKAK is the positive compact operator, it follows from Theorem 2, $(M_3 M_2 M_1)^3$ is the positive compact operator. Moreover, an easy argument shows that

$$\left|(M_3 M_2 M_1)^3 \ (x)\right| \leq \left[(RARAR)^3 + T\right](|x|)$$

for every $x \in E$.

Now consider the non-zero positive operator $S = (RARAR)^3 + T$. Then B and S commute, S dominates the non-zero compact operator $(M_3 M_2 M_1)^3$, and B is quasinilpotent at x_0. By Theorem 3, S and B have a common non-trivial closed invariant ideal. This ideal is invariant under both B and T. □

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