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NONLOCAL CAUCHY PROBLEM FOR NONLINEAR FUNCTIONAL INTEGRODIFFERENTIAL EVOLUTION EQUATIONS IN BANACH SPACES

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Abstract: In this paper, we prove the existence of mild solutions for nonlinear functional integrodifferential evolution equations with nonlocal conditions in Banach spaces. The results are obtained by using the theory of resolvent operators, Banach's contraction principle and Sadovskii's Fixed Point Theorem.

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1. Introduction

In this paper, we study the existence of mild solutions for nonlinear functional integrodifferential evolution equations with nonlocal conditions. More precisely, we consider the following nonlocal Cauchy problem on a general Banach space X:

$$\begin{aligned} &x'(t) = A(t)x(t) + \int_0^t H(t-s)A(s)x(s)ds \\ &\quad + F\left(t, x(\sigma_1(t)), ..., x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(t)))ds\right), \quad t \in J = [0,b], \\ &x(0) + g(x) = x_0, \end{aligned} \tag{1.1}$$

where $A(t)$ is a closed linear operator on X with dense domain $D(A)$ which is

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independent of t and $H(t)$ a bounded operator in X. The nonlinear operators $F : J \times X^{n+1} \to X$, $h : J \times J \times X$, $g : C(J, X) \to X$, $\sigma_i : J \to J, i = 1, ..., n+1$, are given functions.

The study of abstract nonlocal Cauchy problem was motivated by the paper of Byszewski [6], [4]. In [4], the authors have considered the existence and uniqueness of mild, strong, and classical solutions of the nonlocal Cauchy problem, where the operator $A(t) = A$ generates a strongly continuous semigroup. Subsequently, many paper have been interest in the nonlocal Cauchy problem stems mainly from the observation that nonlocal conditions are more realistic than the usual ones in treating physical problems. Byszewski and Akca [5] established the existence of mild and classical solutions of nonlocal Cauchy problem for semilinear functional differential evolution equation. They obtained the results using semigroup of linear operator and Schauder's Fixed Point Theorem. Ntouyas and Tsamatos [14] studied the global existence of solution for semilinear evolution equations with nonlocal conditions by using the Leray-Schauder Alternative. In [13], Lin and Liu discussed the existence of mild solution for autonomous semilinear integrodifferential equations under Lipschitz-type conditions. Fu and Ezzinbi [8] studied the existence of solutions for some neutral functional evolution equations with nonlocal conditions by using the Sadovskii Fixed Point Theorem. Liang et al [11] established some new theorems about the existence of mild solutions for nonlocal Cauchy problems. When $A(t) = A$ is nondensely defined and satisfies the Hille-Yosida condition in Banach space, Benchohra et al [3], Ezzinbi and Liu [7] obtained some results about the existence and regularity of solutions of Cauchy problem with nonlocal conditions. Aizicovici and Lee [1] discussed the existence of integral solutions to the associated nonlocal problem, where $A(t) = A$ is an $m-$accretive operators in X generating a compact nonlinear contraction semigroup. In paper [2], the authors also discussed the existence of integral solutions to the associated nonlocal problem, where $\{A(t)\}_{0 \leq t \leq b}$ is a family of $m-$accretive operators in X generating a compact evolution family. Recently, Liang et al [12] proved the existence of mild solutions of nonlocal Cauchy problems for semilinear nonautonomous evolution equations with compact evolution families on Banach spaces. In this paper, we shall investigate the existence of mild solutions of nonlocal Cauchy problem (1.1) in Banach spaces, by means of a different method, that is, by using the theory of resolvent operators, Banach's contraction principle and Sadovskii's Fixed Point Theorem. The nonlocal Cauchy problems for nonlinear integrodifferential equations with resolvent operators considered here serves as an abstract formulation of partial integrodifferential equations which arises in various applications such as viscoelasticity, heat equations and many other physical phenomena [13], [10], [15].

This paper will be organized as follows. Section 2 gives some preliminaries about theory of resolvent operators which will be used in paper. Section 3 is devoted to the existence of mild solutions of problem (1.1). Further, we present existence results for the nonlocal problem (1.1) for a spacial case. Finally, a concrete example is presented in Section 4 to show the application of our main results.

2. Preliminaries

The purpose of this sections to state some preliminaries about the theory of resolvent operators which are required throughout this paper.

Let X be a Banach space endowed with the norm $\|\cdot\|$, we shall make the following assumptions:

(I) $A(t)$ generates an evolution operator in Banach space X.

(II) Suppose that Y is Banach space formed from $D(A)$ with the graph norm, $H(t) \in B(X), t \in J$. Also $H(t) : Y \to Y$ and for $x(.)$ continuous in Y, $AH(.)x(.) \in L^1([0,b], X)$. For $x \in X, H'(t)x$ is continuous in $t \in J$, where $B(X)$ is the space of all bounded linear operators on X.

Definition 2.1. A resolvent operators for problem (1.1) is a bounded operators valued function $R(t,s) \in B(X), 0 \leq s \leq t \leq b$, the space of bounded linear operators on X, having the following properties:

(a) $R(t,s)$ is strongly continuous in s and t, $R(s,s) = I, 0 \leq s \leq b, \| R(t,s) \| \leq Me^{\beta(t-s)}$ for some constants M and β.

(b) $R(t,s)Y \subset Y, R(t,s)$ is strongly continuous in s and t on Y.

(c) For each $x \in X, R(t,s)x$ is continuously differentiable in $s \in J$ and

$$\frac{\partial R}{\partial s}(t,s)x = -R(t,s)A(s)x - \int_s^t R(t,\tau)H(\tau - s)A(s)xd\tau.$$

(d) For $x \in X$, and $s \in J, R(t,s)x$ is continuously differentiable in $t \in [s,b]$ and

$$\frac{\partial R}{\partial t}(t,s)x = A(t)R(t,s)x + \int_s^t H(t-\tau)A(\tau)R(\tau,s)xd\tau$$

with $\frac{\partial R}{\partial s}(t,s)x$ and $\frac{\partial R}{\partial t}(t,s)x$ are strongly continuous on $0 \leq s \leq t \leq b$. Here, $R(t,s)$ can be extracted from the evolution operator of the generator $A(t)$.

Definition 2.2. A continuous function $x(\cdot) : J \to X$ is a said to be a mild solution to problem (1.1) if for all $x_0 \in X$, it satisfies the following integral equation:

$$\begin{aligned} x(t) &= R(t,0)[x_0 - g(x)] \\ &+ \int_0^t R(t,s)F\left(s, x(\sigma_1(s)), ..., x(\sigma_n(s)), \int_0^s h(s,\tau,x(\sigma_{n+1}(\tau)))d\tau\right)ds. \end{aligned} \quad (2.1)$$

We call P be a condensing operator on a Banach space X, if P continuous and takes bounded sets into bounded sets, and for any bounded subset $D \subseteq X$ with $\alpha(D) \neq 0$, we have $\alpha(P(D)) < \alpha(D)$, where α denotes Kuratowski measure of noncompactness [16].

Lemma 2.1. (Sadovskii's Fixed Point Theorem, see [16]) *Let D be a convex, closed and bounded subset of a Banach space X and $Q : D \to D$ is a condensing map. Then Q has a fixed point in D.*

We remark that a completely continuous operator is the easiest example of a condensing map.

Further we assume the following hypotheses:

(H1) The resolvent operator $R(t,s)$ is compact for $t, s > 0$.

(H2) The function $F : J \times X^{n+1} \to X$ is continuous and there exists constants $L > 0, L_1 > 0$, such that for all $x_i, y_i \in X, i = 1, ...n+1$, we have

$$\| F(t, x_1, x_2, ..., x_{n+1}) - F(t, y_1, y_2, ..., y_{n+1}) \| \leq L\left[\sum_{i=1}^{n+1} \| x_i - y_i \|\right],$$

and

$$L_1 = \max_{t\in J} \| F(t, 0, ..., 0) \| .$$

(H3) The function $h : J \times J \times X \to X$ is continuous and there exists constants $N > 0, N_1 > 0$, such that for all $x, y \in X$,

$$\| h(t,s,x) - h(t,s,y) \| \leq N \| x - y \|,$$

and

$$N_1 = \max_{0\leq s\leq t\leq b} \| h(t,s,0) \| .$$

(H4) $\sigma_i : J \to J, i = 1, ..., n+1$, are continuous functions such that $\sigma_i(t) \leq t, i = 1, ..., n+1$.

(H5) The function $g(\cdot) : C(J,X) \to X$ is continuous and there exists a $\delta \in (0,b)$ such that $g(\phi) = g(\varphi)$ for any $\phi, \psi \in C := C(J,X)$ with $\phi = \psi$ on

$[\delta, b]$.

In addition, there is a continuous nondecreasing function $\Lambda : [0,\infty) \to (0,\infty)$ such that

$$\| g(\phi) \| \leq \Lambda(\| \phi \|), \quad \phi \in C,$$

and for $\rho > 0$

$$\lim_{\rho\to\infty} \sup \frac{\Lambda(\rho)}{\rho} = \gamma < \infty.$$

3. Main Result

Theorem 3.1. *Assume that assumptions* (H1)-(H5) *hold. Also assume that*

$$Me^{[ML(n+Nb)+\beta]b}\gamma < 1. \tag{3.1}$$

Then the nonlocal Cauchy problem (1.1) *has at least one mild solution on* J.

Proof. Consider the space $C := C(J, X)$ the Banach space of all continuous functions from J to X endowed with sup norm.

Let $L_0 := 2ML(n+Nb)+\beta$ and we introduce in the space C the equivalent norm defined as

$$\| \phi \|_V := \sup_{t\in J} e^{-L_0 t} \| \phi(t) \| .$$

Then, it is easy to see that $V := (C(J, X), \| \cdot \|_V)$ is a Banach space. Fix $v \in C$ and for $t \in J, \phi \in V$, we now defined an operator

$$(Q_v\phi)(t) = R(t, 0)[x_0 - g(v)]$$
$$+ \int_0^t R(t, s)F\left(s, x(\sigma_1(s)), ..., x(\sigma_n(s)), \int_0^s h(s, \tau, x(\sigma_{n+1}(\tau)))d\tau\right)ds. \tag{3.2}$$

Since $R(\cdot, 0)(x_0-g(v)) \in C(J, X)$, so, it follows from (H1)-(H4) that $(Q_v\phi)(t) \in V$ for all $\phi \in V$. Let $\phi, \psi \in V$, we have

$$e^{-L_0 t} \| (Q_v\phi)(t) - (Q_v\psi)(t) \|$$
$$\leq e^{-L_0 t} \int_0^t \left\| R(t, s)\left[F\left(s, \phi(\sigma_1(s)), ..., \phi(\sigma_n(s)), \int_0^s h(s, \tau, \phi(\sigma_{n+1}(\tau)))d\tau\right)\right.\right.$$
$$\left.\left. - F\left(s, \psi(\sigma_1(s)), ..., \psi(\sigma_n(s)), \int_0^s h(s, \tau, \psi(\sigma_{n+1}(\tau)))d\tau\right)\right]\right\| ds$$
$$\leq ML \int_0^t e^{-L_0 t} e^{\beta(t-s)}\Big[\| \phi(\sigma_1(s)) - \psi(\sigma_1(s)) \| + \cdots + \| \phi(\sigma_n(s)) - \psi(\sigma_n(s)) \|$$

$$+\left\| \int_0^s h(s,\tau,\phi(\sigma_{n+1}(\tau)))d\tau - \int_0^s h(s,\tau,\psi(\sigma_{n+1}(\tau)))d\tau \right\|\Bigg]ds$$

$$\leq ML\int_0^t e^{-L_0 t}e^{\beta(t-s)}\Bigg[\parallel \phi(s)-\psi(s)\parallel + \cdots + \parallel \phi(s)-\psi(s)\parallel$$

$$+N\int_0^s \parallel \phi(\sigma_{n+1}(\tau))-\psi(\sigma_{n+1}(\tau))\parallel d\tau\Bigg]ds$$

$$\leq ML\int_0^t e^{-L_0 t}e^{\beta(t-s)}\Big[n\parallel \phi(s)-\psi(s)\parallel + Nb\parallel \phi(s)-\psi(s)\parallel\Big]ds$$

$$\leq ML\int_0^t e^{(L_0-\beta)((s-t)}\Big[ne^{-L_0 s}\parallel \phi(s)-\psi(s)\parallel + Nb\sup_{s\in J}e^{-L_0 s}\parallel \phi(s)-\psi(s)\parallel\Big]ds$$

$$\leq ML(n+Nb)\int_0^t e^{(L_0-\beta)(s-t)}ds\parallel \phi-\psi\parallel_V \leq \frac{ML(n+Nb)}{L_0-\beta}\parallel \phi-\psi\parallel_V,\ \ t\in J,$$

which implies that

$$e^{-L_0 t}\parallel (Q_v\phi)(t)-(Q_v\psi)(t)\parallel\leq \frac{1}{2}\parallel \phi-\psi\parallel_V,\quad t\in J.$$

Thus

$$\parallel Q_v\phi - Q_v\psi\parallel_V\leq \frac{1}{2}\parallel \phi-\psi\parallel_V,\quad \phi,\psi\in V.$$

Therefore, Q_v is a strict contraction. By Banach's contraction principle we conclude that Q_v has a unique fixed point $\phi_v\in V$ and equation (3.2) has a unique mild solution on $[0,b]$. Now let $r>0$ and

$$v\in C_r(\delta) := \Big\{\phi\in C([\delta,b],X);\ \sup_{\delta\leq t\leq b}\parallel \phi(t)\parallel\leq r\Big\}.$$

Set

$$\tilde{v}(t) := \begin{cases} v(t) & \text{if } t\in(\delta,b],\\ v(\delta) & \text{if } t\in[0,\delta].\end{cases}$$

From (3.2), we have

$$\phi_{\tilde{v}}(t) = R(t,0)[x_0-g(\tilde{v})] + \int_0^t R(t,s)$$
$$\times F\Big(s,\phi_{\tilde{v}}(\sigma_1(s)),...,\phi_{\tilde{v}}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\Big)ds. \quad (3.3)$$

Then, for $t\in(0,b]$,

$$e^{-\beta t}\parallel \phi_{\tilde{v}}(t)\parallel \leq e^{-\beta t}\parallel R(t,0)[x_0-g(\tilde{v})]\parallel$$

$$+e^{-\beta t}\int_0^t \Big\| R(t,s)F\Big(s,\phi_{\tilde{v}}(\sigma_1(s)),...,\phi_{\tilde{v}}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\Big)\Big\| ds$$

$$\leq M[\| x_0 + g(\tilde{v}) \|]$$
$$+ M \int_0^t e^{-\beta s} \bigg[\bigg\| F\bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau \bigg)$$
$$- F(s,0,...,0) \bigg\| + \| F(s,0,...,0) \| \bigg] ds$$
$$\leq M[\| x_0 \| + \| g(\tilde{v}) \|] + M \int_0^t e^{-\beta s} \bigg\{ L\bigg[\| \phi_{\tilde{v}}(s) \| + \cdots + \| \phi_{\tilde{v}}(s) \|$$
$$+ \int_0^s [\| h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau))) - h(s,\tau,0) \| + \| h(s,\tau,0) \|] d\tau \bigg] + L_1 \bigg\} ds$$
$$\leq M[\| x_0 \| + \Lambda(r)] + M \int_0^t e^{-\beta s} \bigg\{ L\bigg[n \| \phi_{\tilde{v}}(s) \| + b(N \| \phi_{\tilde{v}}(s) \| + N_1) \bigg] + L_1 \bigg\} ds$$
$$\leq M[\| x_0 \| + \Lambda(r)] + MLb(bN_1 + L_1) + ML(n+Nb) \int_0^t \sup_{s\in(0,b]} e^{-\beta s} \| \phi_{\tilde{v}}(s) \| ds.$$

Making use of the Gronwall's inequality, such that

$$\sup_{t\in(0,b]} e^{-\beta t} \| \phi_{\tilde{v}}(t) \| \leq \bigg[M[\| x_0 \| + \Lambda(r)] + MLb(bN_1 + L_1) \bigg] e^{ML(n+Nb)b}.$$

Next we claim that $\phi_{\tilde{v}}(t) \in C_r(\delta)$ for $v \in C_r(\delta)$. If it is not true, then for some positive number r, there is a function $v \in C_r(\delta)$, such that $\phi_{\tilde{v}}(t) \notin C_r(\delta)$, that is, $\| \phi_{\tilde{v}}(t) \| > r$ for some $t \in (0,b]$. Thus

$$1 < \frac{1}{r} \sup_{t\in[\delta,b]} \| \phi_{\tilde{v}}(t) \| \leq \frac{1}{r} M_* + Me^{[ML(n+Nb)+\beta]b} \frac{\Lambda(r)}{r},$$

where M_* is independent of r. Using (H5) and r enough large we conclude that

$$Me^{[ML(n+Nb)+\beta]b}\gamma > 1, \tag{3.4}$$

which is contradicts (3.1). Hence there exists a positive number r, $\phi_{\tilde{v}}(t) \in C_r(\delta)$ for $v \in C_r(\delta)$.

Based on this fact, we will show a mapping $P : C_r(\delta) \to C_r(\delta)$ defined by

$$(Pv)(t) = \phi_{\tilde{v}}(t), \quad t \in [\delta, b]. \tag{3.5}$$

From (3.2) and (H1)-(H4), we deduce that for $v_1, v_2 \in C_r(\delta),\ t \in [0,b]$,

$$e^{-\beta t} \| \phi_{\tilde{v}_1}(t) - \phi_{\tilde{v}_2}(t) \| \leq e^{-\beta t} \| R(t,0)[g(\tilde{v}_1) - g(\tilde{v}_2)] \|$$
$$+ e^{-\beta t} \int_0^t \bigg\| R(t,s) \bigg[F\bigg(s, \phi_{\tilde{v}_1}(\sigma_1(s)), ..., \phi_{\tilde{v}_1}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}_1}(\sigma_{n+1}(\tau)))d\tau \bigg)$$
$$- F\bigg(s, \phi_{\tilde{v}_2}(\sigma_1(s)), ..., \phi_{\tilde{v}_2}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}_2}(\sigma_{n+1}(\tau))) \bigg) \bigg] \bigg\| ds$$

$$\leq M \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel$$

$$+ M \int_0^t Le^{-\beta s}\Big[\parallel \phi_{\tilde{v}_1}(\sigma_1(s)) - \phi_{\tilde{v}_2}(\sigma_1(s)) \parallel + \cdots + \parallel \phi_{\tilde{v}_1}(\sigma_n(s)) - \phi_{\tilde{v}_2}(\sigma_n(s)) \parallel$$

$$+ \left\| \int_0^s h(s,\tau,\phi_{\tilde{v}_1}(\sigma_{n+1}(\tau)))d\tau)) - \int_0^s h(s,\tau,\phi_{\tilde{v}_2}(\sigma_{n+1}(\tau)))d\tau \right\| \Big] ds$$

$$\leq M \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel$$

$$+ ML \int_0^t e^{-\beta s}\Big[\parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel + \cdots + \parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel$$

$$+ N \int_0^s [\parallel \phi_{\tilde{v}_1}(\sigma_{n+1}(s)) - \phi_{\tilde{v}_2}(\sigma_{n+1}(s)) \parallel \Big] ds$$

$$\leq M \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel + ML \int_0^t e^{-\beta s}\Big[n \parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel$$

$$+ Nb \parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel \Big] ds$$

$$\leq M \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel + ML(n+Nb) \int_0^t \sup_{s \in J} e^{-\beta s} \parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel ds.$$

Using again the Gronwall's inequality, that for t, v_1, v_2 as above

$$\sup_{t \in J} e^{-\beta t} \parallel \phi_{\tilde{v}_1}(t) - \phi_{\tilde{v}_2}(t) \parallel \leq M e^{ML(n+Nb)b} \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel,$$

for all $t \in [0, b]$, which implies that

$$\parallel Pv_1 - Pv_2 \parallel \leq M e^{[ML(n+Nb)+\beta]b} \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel,$$

for all $t \in [\delta, b]$, $v_1, v_2 \in C_r(\delta)$. Therefore, P is continuous.

Next we will prove that the P has a fixed point on $C_r(\delta)$, which implies equation (2.1) has an integral solution. To this end, we will show that the family $P(C_r(\delta))$ is a precompact subset of $C_r(\delta)$. Let $\delta \leq t_1 < t_2 \leq b$, we have

$$\parallel Pv(t_2) - Pv(t_1) \parallel$$

$$\leq \parallel [R(t_2,0) - R(t_1,0)][x_0 - g(\tilde{v})] \parallel$$

$$+ \int_0^{t_2} \left\| [R(t_2,s)F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) \right\| ds$$

$$- \int_0^{t_1} \left\| [R(t_1,s)F\left(s, \phi_{\tilde{v}}\sigma_1(s)), ..., \phi_{\tilde{v}}\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) \right\| ds$$

$$\leq \parallel R(t_2,0) - R(t_1,0) \parallel [\parallel x_0 \parallel + \Lambda(r)] + \int_0^{t_1} \parallel R(t_2,s) - R(t_1,s) \parallel$$

$$\left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) \right\| ds + Me^{\beta t_2}$$
$$\times \int_{t_1}^{t_2} e^{-\beta s} \left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}_2}(\sigma_{n+1}(\tau)))d\tau\right) \right\| ds.$$

Noting that

$$\left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) \right\|$$
$$\leq \left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) - F(s,0,...,0) \right\| + \| F(s,0,...,0) \|$$
$$\leq L\left[\| \phi_{\tilde{v}}(\sigma_1(s)) \| + \cdots + \| \phi_{\tilde{v}}(\sigma_n(s)) \| + \left\| \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau \right\| \right] + L_1$$
$$\leq L\left[\| \phi_{\tilde{v}}(s) \| + \cdots + \| \phi_{\tilde{v}}(s) \| + \int_0^s [\| h(s,\tau,\phi_{\tilde{v}}(\tau)) - h(s,\tau,0) \| + \| h(s,\tau,0) \|]d\tau \right] + L_1$$
$$\leq L\left[n \| \phi_{\tilde{v}}(s) \| + b[N \sup_{s\in[\delta,b]} \| \phi_{\tilde{v}}(s) \| + N_1] \right] + L_1$$
$$\leq L\left[(n+Nb) \sup_{s\in[\delta,b]} \| \phi_{\tilde{v}}(s) \| + bN_1 \right] + L_1 \leq L[(n+Nb)r + bN_1] + L_1.$$

We see that $\| Pv(t_2) - Pv(t_1) \|$ tend to zero independently of $v \in C_r(\delta)$ as $t_2 - t_1 \to 0$, since the compactness of $R(t,s)$ for $t,s > 0$, implies the continuity in the uniform operator topology. Thus the family of functions $\{(P_2v) : v \in C_r(\delta)\}$ is equicontinuous on $[\delta, b]$.

It remains to show that $P(C_r(\delta))$ is a precompact subset of $C_r(\delta)$.

Let $\delta < t \leq s \leq b$ be fixed and ε a real number satisfying $0 < \varepsilon < t$, for $v \in C_r(\delta)$, we define

$$(P_\varepsilon v)(t) = R(t,0)[x_0 - g(\tilde{v})]$$
$$+ \int_0^{t-\varepsilon} R(t,s) F\left(s, \phi_{\tilde{v}}\sigma_1(s)), ..., \phi_{\tilde{v}}\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) ds$$

Using the compactness of $R(t,s)$ for $t,s > 0$, we obtain the set $\{(P_\varepsilon v)(t) : v \in Y_r(\delta)\}$ is precompact $v \in C_r(\delta)$ for every $\varepsilon, 0 < \varepsilon < t$. Moreover for every $v \in C_r(\delta)$ we have

$$
\begin{aligned}
&\| (Pv)(t) - (P_\varepsilon v)(t) \| \\
&\le \int_{t-\varepsilon}^{t} \left\| R(t,s)F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) \right\| ds \\
&\qquad \le Me^{\beta b}\int_{t-\varepsilon}^{t} e^{-\beta s}\Bigg[\Bigg\| F\Bigg(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \\
&\qquad\qquad \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\Bigg)\Bigg\|\Bigg] ds \\
&\qquad \le Me^{\beta b}\int_{t-\varepsilon}^{t} e^{-\beta s}\left[L(n+Nb)\sup_{s\in[\delta,b]} \| \phi_{\tilde{v}}(s) \| + LbN_1 + L_1\right] ds \\
&\qquad\qquad \le Me^{\beta b}\int_{t-\varepsilon}^{t} e^{-\beta s}[L(n+Nb)r + LbN_1 + L_1] ds,
\end{aligned}
$$

therefore there are precompact sets arbitrarily close to the set $\{(Pv) : v \in C_r(\delta)\}$. Hence the set $\{(Pv) : v \in C_r(\delta)\}$is a precompact in X. There arguments enable us to conclude that P is completely continuous. We can now apply Sadovskii's Fixed Point Theorem to conclude that P has at least fixed point $\tilde{v}_* \in C_r(\delta)$. Let $x = \phi_{\tilde{v}_*}$. Then, we have

$$
\begin{aligned}
&x(t) = R(t,0)[x_0 - g(\tilde{v}_*)] \\
&\quad + \int_0^t R(t,s)F\left(s, x(\sigma_1(s)), ..., x(\sigma_n(s)), \int_0^s h(s,\tau,x(\sigma_{n+1}(\tau)))d\tau\right) ds. \qquad (3.6)
\end{aligned}
$$

Noting that $x = \phi_{\tilde{v}_*} = (P\tilde{v}_*)(t) = \tilde{v}_*$, $t \in [\delta, b]$. By (H4), we obtain

$$g(x) = g(\tilde{v}_*).$$

This implies that x is Q has a fixed point in $C_r(\delta) \subset C(J,X)$. Hence, problem (1.1) has a mild solution and completes the proof of Theorem 3.1.

Remark 3.1. As we all know, most of work discuss related semilinear nonlocal Cauchy problem when g satisfy Lipschitz-type conditions, or convex and compact on a given ball [5], [14], [8], [3], [7]. In this paper, we consider the case g is continuous but without imposing severe compactness conditions and convexity.

Remark 3.2. Condition (H5) on g in the above theorem is an extension of the corresponding conditions in paper [11], [1].

4. Application

To illustrate the application of the obtained results of this paper, we study the following example in this section:

$$\begin{aligned}
&z_t(t,x) = \frac{\partial^2}{\partial x^2}\left[a_0(t)z(t,x) + \int_0^t l(t-s)z(s,x)ds\right] \\
&+ a_1(t)z(t,x) + \sin z(t,x) + \frac{1}{1+t^2}\int_0^t a_2(s)z(s,x)ds, \\
&z(t,0) = z(t,\pi) = 0, \\
&z(0,x) + \int_\delta^1 [z(s,x) + \log(1+|z(s,x)|)]ds = z_0(x), \\
&0 \le t \le 1, \quad 0 \le x \le \pi,
\end{aligned} \tag{4.1}$$

where $\delta > 0$, $z_0(x) \in X = L^2([0,\pi])$ and $z_0(0) = z_0(\pi) = 0$. Here, the functions $a_0(t)$ and $l(t)$ is continuous on J.

Let $X = L^2([0,\pi])$ and the operators $A(t)$ be defined by

$$A(t)w = a_0(t)w''$$

with the domain $D(A) = \{w \in X : w, w''$ are absolutely continuous, $w'' \in X, w(0) = w(1) = 0\}$, then $A(t)$ generates an evolution system and $R(t,s)$ can be extracts from the evolution systems [15], [9] such that $R(t,s)$ is compact and $\| R(t,s) \| \le Me^{\beta(t-s)}$ for some constants M and β.

We assume that the function $a_i(\cdot)$ is continuous on $[0,1]$, and

$$l_i = \sup_{0\le s\le 1} |a_i(s)| < 1, \quad i = 1,2.$$

Define respectively $F : [0,1] \times X \times X \to X, h : [0,1] \times [0,1] \times X \to X$ and $g : C([0,1],X) \to X$ by

$$\begin{aligned}
&F\left(t, z, \int_0^t h(t,s,z(s))ds\right)(x) \\
&\qquad = a_1(t)z(t,x) + \sin z(t,x) + \frac{1}{1+t^2}\int_0^t a_2(s)z(s,x)ds,
\end{aligned}$$

$$\int_0^t h(t,s,z(s))(x)ds = \frac{1}{1+t^2}\int_0^t a_2(s)z(s,x)ds,$$

and

$$g(z)(x) = \int_\delta^1 [z(s,x) + \log(1+|z(s,x)|)]ds, \quad x \in C([0,1],X).$$

Then equation (4.1) takes the abstract form (1.1). If we assume that

$Me^{M(1+l_1+l_2)(1+l_1)+\beta}(1-\delta) < 1$. From Theorem 3.1, we deduce that nonlocal Cauchy problem (4.1) has a mild solution on $[0,1]$.

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