International Journal of Pure and Applied Mathematics
Volume 50 No. 4 2009, 497-503

SOME REFINEMENTS OF HARDY'S INEQUALITY

Gong Yan[1], Gao Mingzhe[2 §]
[1,2]Department of Mathematics and Computer Science
Normal College
Jishou University
Jishou, Hunan, 416000, P.R. CHINA
[1]e-mail: hnxxjdsysjx@163.com
[2]e-mail: mingzhegao@163.com

Abstract: In this paper it is shown that the Hardy inequality for discrete form can be improved by means of a sharpening of Hölder's inequality. A similar result for the Hardy integral inequality is also proved. And the coefficient $\left(\frac{p}{p-1}\right)^p$ of the classical Hardy inequality is further discussed.

AMS Subject Classification: 26D15
Key Words: Hardy's inequality, Hölder's inequality, parametric variable vector, integration by parts

1. Introduction

Let $a_n \geq 0$, $\beta_n = \frac{1}{n}\sum_{k=1}^{n} a_k$, $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. Then

$$\sum_{n=1}^{\infty} \beta_n^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p\,, \tag{1}$$

where the constant factor $\left(\frac{p}{p-1}\right)^p$ is best possible. This is famous Hardy's inequality (see [3]). The correspondent integral form of it is that

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx\,, \tag{2}$$

where $f(x) \geq 0$ and $F(x) = \int_0^x f(t)dt$, and the constant factor $\left(\frac{p}{p-1}\right)^p$ contained in (2) is also best possible. Because the applications of the inequalities

Received: February 20, 2008

§Correspondence author

(1) and (2) are very important in analysis and others, they have been studyed and generalized by mathematicians (such as [4], [1], [2], etc.). The purpose of this paper is to build a new inequality which is a sharpening of Hölder's inequality and then to realize significant improvements of the inequalities (1) and (2).

For convenience, we firstly introduce some notations:

$$(a^r, b^s) = \sum_{n=1}^{\infty} a_n^r b_n^s, \quad \|a\|_p = \left(\sum_{n=1}^{\infty} a_n^p \right)^{1/p} \quad \text{and} \quad \|a\|_2 = \|a\|;$$

$$(f^r, g^s) = \int_0^{+\infty} f^r(x) g^s(x) dx, \ \|f\|_p = \left(\int_0^{\infty} f^p(x) dx \right)^{1/p} \text{ and } \|f\|_2 = \|f\|.$$

We next introduce a function defined by

$$S_p(\alpha, x) = (\alpha^{p/2}, x) \|\alpha\|_p^{-p/2},$$

where x is a parametric variable vector which is a variable unit vector. Under general case, it is chosen properly such that the specific problems discussed are simplified. Clearly, $S_p(\alpha, x) = 0$ when the vector x selected is orthogonal to $\alpha^{p/2}$.

Throughout this paper, the exponent m indicates that $m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

2. Lemmas and their Proofs

In order to verify our assertions, we need to build the following lemmas.

Lemma 1. *Let $a_n, b_n \geq 0, \frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|a\|_p < +\infty$ and $0 < \|b\|_q < +\infty$, then*

$$(a, b) \leq \|a\|_p \|b\|_q (1 - r)^m, \tag{3}$$

where $r = (S_p(a, c) - S_q(b, c))^2, \|c\| = 1$ and $(a^{p/2}, c)(b^{q/2}, c) \geq 0$. And the equality in (3) holds if and only if $a^{p/2}$ and $b^{q/2}$ are linearly dependent; or c is a linear combination of $a^{p/2}$ and $b^{q/2}$, and $(a^{p/2}, c) = 0$ or $(b^{q/2}, c) = 0$.

Proof. Without loss of generality, suppose that $p > q > 1$. Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p > 2$. Putting $R = \frac{p}{2}, Q = \frac{p}{p-2}$, then $\frac{1}{R} + \frac{1}{Q} = 1$. By Hölder's inequality we obtain

$$(a, b) = \sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} (a_k b_k^{q/p}) b_k^{1-q/p}$$

$$\leq \left(\sum_{k=1}^{\infty}(a_k b_k^{q/p})^R\right)^{\frac{1}{R}} \left(\sum_{k=1}^{\infty}(b_k^{1-q/p})^Q\right)^{\frac{1}{Q}} = (a^{p/2}, b^{q/2})^{2/p}\|b\|_q^{q(1-2/p)}. \quad (4)$$

And the equality in (4) holds if and only if $a^{p/2}$ and $b^{q/2}$ are linearly dependent. In fact, the equality in (4) holds if and only if to any k, there exists c_1 such that $(a_k b_k^{q/p})^R = c_1(b_k^{1-q/p})^Q$. It is easy to deduce that $b_k^{q/2} = c_0 a_k^{q/2}$, where $c_0 = c_1^{2/q}$.

In our previous paper [5], with the help of the positive definiteness of Gram matrix it established an important inequality of the form

$$(\alpha, \beta)^2 \leq \|\alpha\|^2\|\beta\|^2 - (\|\alpha\|x - \|\beta\|y)^2 , \quad (5)$$

where $x = (\beta, \gamma), y = (\alpha, \gamma), \|\gamma\| = 1$ and $xy \geq 0$. And the equality in (5) holds if and only if α and β are linearly dependent; or the vector γ is a linear combination of α and β, and $x = 0$ or $y = 0$. If α, β and γ in (5) are replaced by $a^{p/2}, b^{q/2}$ and c respectively, then we get

$$(a^{p/2}, b^{q/2})^2 \leq \|a\|_p^p\|b\|_q^q(1 - r) , \quad (6)$$

where $r = (S_p(a, c) - S_q(b, c))^2$. And the equality in (6) holds if and only if $a^{p/2}$ and $b^{q/2}$ are linearly dependent; or c is a linear combination of $a^{p/2}$ and $b^{q/2}$, and $(a^{p/2}, c) = 0$ or $(b^{q/2}, c) = 0$. Substituting (6) into (4) we obtain after simplifications

$$(a, b) \leq \|a\|_p\|b\|_q(1 - r)^{1/p} . \quad (7)$$

Noticing the symmetry of p and q, the inequality (3) follows from (7). The proof of the lemma is completed. □

When $p = 2$, basing on (5) we attain the following result at once.

Corollary 1. *If $0 < \|a\| < +\infty$ and $0 < \|b\| < +\infty$, then*

$$(a, b) \leq \|a\|\|b\|(1 - \tilde{r})^{1/2} , \quad (8)$$

where $\tilde{r} = \left(\frac{(a,c)}{\|a\|} - \frac{(b,c)}{\|b\|}\right)^2, \|c\| = 1$ and $(a, c)(b, c) \geq 0$. And the equality in (8) holds if and only if a and b are linearly dependent, or c is a linear combination of a and b, and $(a, c) = 0$ or $(b, c) = 0$.

Lemma 2. *Let $f(x), g(x) \geq 0, \frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|f\|_p < +\infty$ and $0 < \|g\|_q < +\infty$, then*

$$(f, g) \leq \|f\|_p\|g\|_q(1 - R)^m , \quad (9)$$

where $R = (S_p(f, h) - S_q(g, h))^2, (f^{p/2}, h)(g^{q/2}, h) \geq 0$ and $\|h\| = 1$. And the equality in (9) holds if and only if $f^{p/2}(x)$ and $g^{q/2}(x)$ are linearly dependent or $h(x)$ is a linear combination of $f^{p/2}(x)$ and $g^{q/2}(x)$, and $(f^{p/2}, h) = 0$ or $(g^{q/2}, h) = 0$.

Its proof is similar to that of Lemma 1, hence it is omitted here.

Corollary 2. *If* $0 < \|f\| < +\infty$ *and* $0 < \|g\| < +\infty$, *then*

$$(f,g) \le \|f\|\|g\| \left\{ 1 - \left(\frac{(g,h)}{\|g\|} - \frac{(f,h)}{\|f\|} \right)^2 \right\}^{1/2}, \tag{10}$$

where $(g,h)(f,h) \ge 0$. *And the equality in (10) holds if and only if* f *and* g *are linearly dependent; or* h *is a linear combination of* f *and* g, *and* $(f,h) = 0$ *or* $(g,h) = 0$.

3. Theorems and their Corollaries

In this section we shall deal with Hardy's inequality with the help of the above results.

Theorem 1. *Let* $a_1 > 0, a_n \ge 0 \quad (n = 2,3,\cdots), \beta_n = \frac{1}{n}\sum_{k=1}^{n} a_k, \frac{1}{p} + \frac{1}{q} = 1$ *and* $p > 1$. *If* $\|a\|_p < +\infty$, *then*

$$\|\beta\|_p < \left(\frac{p}{p-1} \right) (1-r)^m \|a\|_p \,, \tag{11}$$

where $r = a_1^p \left(\|a\|_p^{-p/2} - \|\beta\|_p^{-p/2} \right)^2$.

Proof. We apply Hardy's techniques to estimate $\sum_{n=1}^{N} \beta_n^p$. At the first we estimate the difference of the following two terms:

$$\begin{aligned} \beta_n^p - \frac{p}{p-1}\beta_n^{p-1}a_n &= \beta_n^p - \frac{p}{p-1}(n\beta_n - (n-1)\beta_{n-1})\beta_n^{p-1} \\ &= \beta_n^p \left(1 - \frac{np}{p-1}\right) + \frac{(n-1)p}{p-1}\beta_n^{p-1}\beta_{n-1} \\ &= \beta_n^p \left(1 - \frac{np}{p-1}\right) + \frac{(n-1)p}{p-1}\left((\beta_n^p)^{p-1}(\beta_{n-1}^p)\right)^{1/p}. \end{aligned} \tag{12}$$

Applying arithmetic geometric mean inequality to the second term of the right-hand side of (12) we get

$$\left((\beta_n^p)^{p-1}(\beta_{n-1}^p)\right)^{\frac{1}{p}} \le \frac{1}{p}((p-1)\beta_n^p + \beta_{n-1}^p) \tag{13}$$

It follows from (12) and (13) that

$$\beta_n^p - \frac{p}{p-1}\beta_n^{p-1}a_n \quad \le \quad \beta_n^p \left(1 - \frac{np}{p-1}\right) + \frac{n-1}{p-1}\left((p-1)\beta_n^p + \beta_{n-1}^p\right)$$

$$= \frac{1}{p-1}\left((n-1)\beta_{n-1}^p - n\beta_n^p\right).$$

Summing of the above inequality with respect to n, we have

$$\sum_{n=1}^{N}\beta_n^p - \frac{p}{p-1}\sum_{n=1}^{N}\beta_n^{p-1}a_n \le -\frac{1}{p-1}(N\beta_N^p) \le 0.$$

Whence $\sum_{n=1}^{N}\beta_n^p \le \frac{p}{p-1}\sum\limits_{n=1}^{N}\beta_n^{p-1}a_n$. Let $N\to\infty$. We attain that

$$\sum_{n=1}^{\infty}\beta_n^p \le \frac{p}{p-1}\sum_{n=1}^{\infty}\beta_n^{p-1}a_n. \tag{14}$$

Applying the inequality (3) to the right-hand side of (14) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\beta_n^p &\le \frac{p}{p-1}\sum_{n=1}^{\infty}a_n\beta_n^{p-1} \le \frac{p}{p-1}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\beta_n^{(p-1)q}\right)^{\frac{1}{q}}(1-r)^m\\ &= \frac{p}{p-1}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\beta_n^p\right)^{\frac{1}{q}}(1-r)^m.\end{aligned}$$

Hence

$$\|\beta\|_p \le \frac{p}{p-1}(1-r)^m\|a\|_p. \tag{15}$$

It remains to define only r in (15). According to the inequality (3), the real number r is defined by

$$r = (S_p(a,c) - S_q(\beta^{p-1},c))^2. \tag{16}$$

We choose a sequence $c = \{c_n | c_1 = 1, c_n = 0, n = 2,3,\cdots\}$.

Clearly, $\|c\| = 1$. It is easy to deduce that $(a^{p/2},c) = a_1^{p/2}, (\beta^{(p-1)q/2},c) = a_1^{p/2}$ and $\|\beta^{p-1}\|_q^{-q/2} = \|\beta\|_p^{-p/2}$. We obtain therefore from (16)

$$r = a_1^p(\|a\|_p^{-p/2} - \|\beta\|_p^{-p/2})^2. \tag{17}$$

Since a, β^{p-1} and c are linearly independent, it is impossible to have equality in (15). The theorem is proved. □

Remark. If the real number r in (11) is replaced by zero, then the inequality (1) is yielded. We find from (11) that $r \ne 0$ provided that $\|a\|_p \ne \|\beta\|_p$. In other words, if $\|a\|_p$ and p are finite and $\|a\|_p \ne \|\beta\|_p$, then the constant $\left(\frac{p}{p-1}\right)^p$ in (1) is not best possible. And the inequality (11) is obviously an improvement of (1).

In particular, when $p = 2$, basing on (8) and (11) we have the following result.

Corollary 3. *Let* $a_1 > 0, a_n \geq 0 (n = 2, 3, \cdots)$ *and* $\beta_n = \frac{1}{n}\sum_{k=1}^{n} a_k$. *If* $\|a\| < +\infty$ *then*

$$\|\beta\| < 2(1-\bar{r})^{1/2}\|a\|\,, \tag{18}$$

where $\tilde{r} = a_1^2\left(\frac{1}{\|a\|} - \frac{1}{\|\beta\|}\right)^2$.

Theorem 2. *Let* $f(x) \geq 0, g(x) = \frac{1}{x}\int_0^x f(t)dt, \frac{1}{p} + \frac{1}{q} = 1$ *and* $p > 1$. *If* $0 < \int_0^\infty f^p(t)dt < +\infty$, *then*

$$\|g\|_p < \frac{p}{p-1}(1-R)^m\|f\|_p\,, \tag{19}$$

where $R = (A\|f\|_p^{-p/2} - B\|g\|_p^{-p/2})^2$. *Here* A *and* B *are defined by* $A = \int_0^1 f^{p/2}(x)dx$ *and* $B = \int_0^1 g^{p/2}(x)dx$.

Proof. Using integration by parts we attain that

$$\|g\|_p^p = \int_0^\infty g^p(x)dx = \frac{p}{p-1}(f, g^{p-1})\,. \tag{20}$$

Applying the inequality (9) to estimate the inner product of the right-hand side of (20) as follows:

$$(f, g^{p-1}) \leq \|f\|_p\|g^{p-1}\|_q(1-R)^m = \|f\|_p\|g\|_p^{p-1}(1-R)^m\,. \tag{21}$$

Substituting (21) into (20) we obtain after simplifications

$$\|g\|_p \leq \left(\frac{p}{p-1}\right)(1-R)^m\|f\|_p. \tag{22}$$

It remains to define R in (22). Basing on the inequality (9) we have

$$R = (S_p(f,h) - S_q(g^{p-1},h))^2.$$

We choose a function h such that

$$h(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & x > 0. \end{cases}$$

Clearly, $\|h\| = 1$. It is easy to deduce that

$$S_p(f,h) = \|f\|_p^{-p/2}(f^{p/2},h) = \|f\|_p^{-p/2}\int_0^1 f^{p/2}(x)dx$$

and $S_q(g^{p-1},h) = \|g^{p-1}\|_q^{-q/2}(g^{(p-1)q/2},h) = \|g\|_p^{-p/2}\int_0^1 g^{p/2}(x)dx$.

Since f, g^{p-1} and h are linearly independent, it is impossible to have equality in (22). Thus the proof of the theorem is completed. □

Remark. If R is (19) is replaced by zero, then the inequality (2) is yielded. Being similar to the remark given at the end of the proof of Theorem 1, we find from (19) that if $\|f\|_p$ and p are finite, then $R \neq 0$. Under such case, the constant factor $\left(\frac{p}{p-1}\right)^p$ in (2) is also not best possible. And the inequality (19) is obviously an improvement of (2).

When $p = 2$, we get from (19) and (10) the following result.

Corollary 4. *Let* $f(x) \geq 0, g(x) = \frac{1}{x}\int_0^x f(t)dt$. *If* $\|f\| < +\infty$, *then*

$$\|g\| < 2(1-\tilde{R})^{1/2}\|f\|,$$

where $\tilde{R} = \left(\|f\|^{-1}\int_0^1 f(t)dt - \|g\|^{-1}\int_0^1 g(t)dt\right)^2$.

Acknowledgements

Authors express their thanks to Professor L.G. Hsu for his encouragement.

Foundation item: A Project Supported by scientific Research Fund of Hunan Provincial Education Department (06C657).

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