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ABOUT SOME BIVARIATE OPERATORS OF SCHURER-STANCU TYPE

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Abstract: In this paper, we will obtain a form of Schurer-Stancu bivariate operators and finally we will give an approximation theorem for them.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\Delta_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x, y \geq 0, x + y \leq 1\}$. For $m \in \mathbb{N}$, the operator B_m defined for any function $f \in C(\Delta_2)$ by

$$(B_m f)(x, y) = \sum_{\substack{k,j=0 \\ k+j \leq m}} p_{m,k,j}(x, y) f\left(\frac{k}{m}, \frac{j}{m}\right) \tag{1.1}$$

for any $(x, y) \in \Delta_2$, where

$$p_{m,k,j}(x, y) = \frac{m!}{k!j!(m-k-j)!} x^k y^j (1-x-y)^{m-k-j}, \tag{1.2}$$

for any $k, j \in \mathbb{N}_0$, $k + j \leq m$ and any $(x, y) \in \Delta_2$ is named the Bernstein bivariate operator (see [12]).

Let $p \in \mathbb{N}_0$ be given and $m \in \mathbb{N}$. The operator $\tilde{B}_{m,p}$ defined for any function $f \in C([0, 1 + p]^2)$ and any $(x, y) \in \Delta_2$ by

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$$(\tilde{B}_{m,p}f)(x,y) = \sum_{\substack{k,j=0\\k+j\le m+p}} p_{m+p,k,j}(x,y) f\left(\frac{k}{m},\frac{j}{m}\right), \tag{1.3}$$

is a bivariate operator of Schurer type (see [10]).

Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be given real parameters such that $0 \le \alpha_1 \le \beta_1$, $0 \le \alpha_2 \le \beta_2$. For $m \in \mathbb{N}$, the operator $S_m^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}$, defined for any function $f \in C([0,1]^2)$ by

$$(S_m^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}f)(x,y) = \sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y) f\left(\frac{k+\alpha_1}{m+\beta_1},\frac{j+\alpha_2}{m+\beta_2}\right), \tag{1.4}$$

for any $(x,y) \in \Delta_2$ is a bivariate operator of Stancu type (see [11]).

Let $e_{ij} : [0,1+p]^2 \to \mathbb{R}$ be the functions test, defined by $e_{ij}(x,y) = x^i y^j$ for any $(x,y) \in [0,1+p]^2$, where $i,j \in \mathbb{N}_0$. In the paper [9] the following representation for the polynomials $B_m e_{ij}$ is proved.

Lemma 1.1. *The operators $(B_m)_{m\ge1}$ verify for any $(x,y) \in \Delta_2$ and any $m \in \mathbb{N}$, $i,j \in \mathbb{N}_0$ the following equality:*

$$(B_m e_{ij})(x,y) = \frac{1}{m^{i+j}} \sum_{\nu=0}^{i} \sum_{\mu=0}^{j} m^{[\nu+\mu]} S(i,\nu) S(j,\mu) x^\nu y^\mu, \tag{1.5}$$

where $S(i,\nu)$, $S(j,\mu)$ are the Stirling's numbers of second kind and $m^{[k]} = m(m-1)\dots(m-k+1)$, $k \in \mathbb{N}_0$, $m^{[0]} = 1$.

Let $f : [0,1+p]^2 \to \mathbb{R}$ be a bounded function. The function $\omega_{total}(f;\cdot,\cdot) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined for any $(\delta_1,\delta_2) \in [0,\infty) \times [0,\infty)$ by

$$\omega_{total}(f;\delta_1,\delta_2) = \sup\left\{|f(x,y) - f(x',y')| : (x,y),(x',y') \in [0,1+p]^2, \; |x-x'| \le \delta_1, |y-y'| \le \delta_2\right\} \tag{1.6}$$

is called the first order modulus of smoothness of function f or total modulus of continuity of function f. For some further information on this measure of smoothness see for example [5] or [16]. The following result is given in [15].

Theorem 1.1. *Let $L : C([0,1+p]^2) \to B([0,1+p]^2)$ be a constant reproducing linear positive operator. For any $f \in C([0,1+p]^2)$, any $(x,y) \in [0,1+p]^2$ and any $\delta_1, \delta_2 > 0$, the following inequality*

$$|(Lf)(x,y) - f(x,y)| \le \left(1 + \delta_1^{-1}\sqrt{(L(\cdot - x)^2)(x,y)}\right) \cdot \left(1 + \delta_2^{-1}\sqrt{(L(* - y)^2)(x,y)}\right) \omega_{total}(f;\delta_1,\delta_2) \tag{1.7}$$

holds, where "$\cdot$" and "$*$" stand for the first and the second variable.

The purpose of this paper is to give a representation for the bivariate operators and GBS operators of Schurer-Stancu type, to establish a convergence theorem for these operators. We also give an approximation theorem for these operators in terms of the first modulus of smoothness and of the mixed modulus of smoothness.

2. The Construct of the Bivariate Operators of Schurer-Stancu Type. Approximation and Convergence Theorems

In the paper [5], the following construct is given. Let $p_1, p_2 \in \mathbb{N}$ be given and let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be real parameters satisfying the conditions $0 \leq \alpha_1 \leq \beta_1$ and $0 \leq \alpha_2 \leq \beta_2$. For $m_1, m_2 \in \mathbb{N}$, the operator $\tilde{S}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{m_1,m_2,p_1,p_2} : C([0,1+p_1] \times [0,1+p_2]) \to C([0,1] \times [0,1])$, defined for any function $f \in C([0,1+p_1] \times [0,1+p_2])$ and any $(x,y) \in [0,1+p_1] \times [0,1+p_2]$ by

$$(\tilde{S}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{m_1,m_2,p_1,p_2} f)(x,y) = \sum_{k_1=0}^{m_1+p_1} \sum_{k_2=0}^{m_2+p_2} p_{m_1+p_1,k_1}(x) p_{m_2+p_2,k_2}(y) \cdot \cdot f\left(\frac{k_1+\alpha_1}{m_1+\beta_1}, \frac{k_2+\alpha_2}{m_2+\beta_2}\right), \quad (2.1)$$

where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$, $m \in \mathbb{N}$, $k \in \{0,1,\ldots,m\}$ and $x \in [0,1]$ are the Bernstein fundamental polynomials, is called the Schurer-Stancu bivariate operator. Some properties of this operator are given there, too.

Next, we will construct a type of Schurer-Stancu bivariate operator, inspired by the Bernstein bivariate operator (1.1). Let $p \in \mathbb{N}_0$ be given and let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be given real parameters such that $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$. For $m \in \mathbb{N}$, the operator $\tilde{S}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{m,p}$, defined for any function $f \in C([0,1+p]^2)$ by

$$(\tilde{S}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{m,p} f)(x,y) = \sum_{\substack{k,j=0 \\ k+j \leq m+p}} p_{m+p,k,j}(x,y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_1}{m+\beta_1}\right), \quad (2.2)$$

for any $(x,y) \in \Delta_2$ is an operator of Schurer-Stancu type. Obviously, this operator is linear and positive on Δ_2. For $\beta_1 = \beta_2 = 0$, we obtain the bivariate operator of Schurer type (1.3), for $p = 0$, we obtain the bivariate operator of Stancu type (1.4) and for $p = \beta_1 = \beta_2 = 0$, we obtain the bivariate operator of Bernstein type (1.1).

Lemma 2.1. *The operators* $(\tilde{S}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}_{m,p})_{m \geq 1}$ *verify for any* $(x,y) \in \Delta_2$

the following equalities:

$$(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{00})(x,y)=1, \tag{2.3}$$

$$(m+\beta_1)(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{10})(x,y)=(m+p)x+\alpha_1, \tag{2.4}$$

$$(m+\beta_2)(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{01})(x,y)=(m+p)y+\alpha_2, \tag{2.5}$$

$$(m+\beta_1)^2(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{20})(x,y)=(m+p)(m+p-1)x^2 + (1+2\alpha_1)(m+p)x+\alpha_1^2 \tag{2.6}$$

and

$$(m+\beta_2)^2(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{02})(x,y)=(m+p)(m+p-1)y^2 + (1+2\alpha_2)(m+p)y+\alpha_2^2. \tag{2.7}$$

Proof. We use the equalities

$$(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{00})(x,y)=(B_{m+p}e_{00})(x,y),$$

$$(m+\beta_1)(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{10})(x,y)=m(B_{m+p}e_{10})(x,y)+\alpha_1(B_{m+p}e_{00})(x,y),$$

$$(m+\beta_2)(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{01})(x,y)=m(B_{m+p}e_{01})(x,y)+\alpha_2(B_{m+p}e_{00})(x,y),$$

$$(m+\beta_1)^2(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{20})(x,y)=m^2(B_{m+p}e_{20})(x,y)+2\alpha_1 m\cdot \cdot(B_{m+p}e_{10})(x,y)+\alpha_1^2(B_{m+p}e_{00})(x,y)$$

and

$$(m+\beta_2)^2(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{02})(x,y)=m^2(B_{m+p}e_{02})(x,y)+2\alpha_2 m\cdot \cdot(B_{m+p}e_{01})(x,y)+\alpha_2^2(B_{m+p}e_{00})(x,y). \qquad \square$$

Lemma 2.2. *The operators $(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)})_{m\geq 1}$ verify for any $(x,y)\in\Delta_2$ the following equalities:*

$$(m+\beta_1)^2(\tilde{S}_{m,p}^{(\alpha,\beta)}(\cdot-x)^2)(x,y)=(m+p)x(1-x)+((p-\beta_1)x+\alpha_1)^2 \tag{2.8}$$

and

$$(m+\beta_2)^2(\tilde{S}_{m,p}^{(\alpha,\beta)}(*-y)^2)(x,y)=(m+p)y(1-y)+((p-\beta_2)y+\alpha_2)^2. \tag{2.9}$$

Proof. We use the equalities (2.3)-(2.7) and the relations

$$(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot-x)^2)(x,y)=(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{20})(x,y) -2x(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{10})(x,y)+x^2(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{00})(x,y)$$

and

$$(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(*-y)^2)(x,y)=(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{02})(x,y)$$

$$- 2y(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{01})(x,y) + y^2(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{00})(x,y). \quad \square$$

Lemma 2.3. *The operators* $(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)})_{m\geq 1}$ *verify for any* $(x,y) \in \Delta_2$ *the following inequalities:*

$$4(m+\beta_1)^2(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot - x)^2)(x,y) \leq m + p + 4\gamma_1 \tag{2.10}$$

and

$$4(m+\beta_2)^2(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(* - y)^2)(x,y) \leq m + p + 4\gamma_2, \tag{2.11}$$

where $\gamma_1 = \max\{\alpha_1^2, (p - \beta_1 + \alpha_1)^2\}$ *and* $\gamma_2 = \max\{\alpha_2^2, (p - \beta_2 + \alpha_2)^2\}$.

Proof. We use the relations (2.8), (2.9) and the inequalities $x(1-x) \leq 1/4$, $y(1-y) \leq 1/4$, $((p-\beta_1)x + \alpha_1)^2 \leq \gamma_1$, $((p-\beta_2)y + \alpha)^2 \leq \gamma_2$, for any $x, y \in [0,1]$. $\square$

Theorem 2.1. *If* $f \in C([0,1+p]^2)$, *then for any* $(x,y) \in \Delta_2$ *and any* $m \in \mathbb{N}$, *we have the following inequalities:*

$$|f(x,y) - (\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}f)(x,y)| \leq \left(1 + \delta_1^{-1}\sqrt{\frac{m+p+4\gamma_1}{4(m+\beta_1)^2}}\right)$$
$$\times \left(1 + \delta_2^{-1}\sqrt{\frac{m+p+4\gamma_2}{4(m+\beta_2)^2}}\right)\omega_{total}(f;\delta_1,\delta_2), \tag{2.12}$$

for any $\delta_1, \delta_2 > 0$ *and*

$$|f(x,y) - (\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}f)(x,y)|$$
$$\leq 4\omega_{total}\left(f; \sqrt{\frac{m+p+4\gamma_1}{4(m+\beta_1)^2}}, \sqrt{\frac{m+p+4\gamma_2}{4(m+\beta_2)^2}}\right).$$

Proof. The relation (2.12) results from Theorem 1.1 and Lemma 2.2; choosing by $\delta_1 = \sqrt{\frac{m+p+4\gamma_1}{4(m+\beta_1)^2}}$ and $\delta_2 = \sqrt{\frac{m+p+4\gamma_2}{4(m+\beta_2)^2}}$, we obtain the relation (2.13). $\square$

Corollary 2.1. *If* $f \in C([0,1+p]^2)$, *then*

$$\lim_{m\to\infty} \tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}f = f \tag{2.13}$$

uniformly on Δ_2.

3. Approximation and Convergence Theorems for GBS Operators of Schurer Type

A function $f:[0,1+p]^2\to\mathbb{R}$ is called B-continuous (Bögel-continuous) function at $(x_0,y_0)\in[0,1+p]^2$ if

$$\lim_{(x,y)\to(x_0,y_0)}\Delta f\left[(x,y),(x_0,y_0)\right]=0.$$

Here $\Delta f\left[(x,y),(x_0,y_0)\right]=f(x,y)-f(x_0,y)-f(x,y_0)+f(x_0,y_0)$ denotes a so-called mixed difference of f.

A function $f:[0,1+p]^2\to\mathbb{R}$ is called B-differentiable (Bögel-differentiable) function at $(x_0,y_0)\in[0,1+p]^2$ if it exists and if the limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)}\frac{\Delta f\left[(x,y),(x_0,y_0)\right]}{(x-x_0)(y-y_0)}.$$

The limit is named the B-differential of f at the point (x_0,y_0) and is noted by $D_Bf(x_0,y_0)$.

The definitions of B-continuity and B-differentiability were introduced by K. Bögel in the papers [7] and [8].

The function $f:[0,1+p]^2\to\mathbb{R}$ is B-bounded on $\Delta_{2,p}$ if there exists $K>0$ such that

$$\left|\Delta f\left[(x,y),(s,t)\right]\right|\le K$$

for any $(x,y),(s,t)\in[0,1+p]^2$.

We shall use the function sets $B([0,1+p]^2)=\left\{f:[0,1+p]^2\to\mathbb{R}|f \text{ bounded on } [0,1+p]^2\right\}$ with the usual sup-norm $\|\cdot\|_\infty$, $B_b([0,1+p]^2)=\left\{f:[0,1+p]^2\to\mathbb{R}|f\ B\text{-bounded on } [0,1+p]^2\right\}$ and we set

$$\|f\|_B=\sup_{(x,y),(s,t)\in[0,1+p]^2}\left|\Delta f\left[(x,y),(s,t)\right]\right|,$$

where $f\in B_b([0,1+p]^2)$, $C_b([0,1+p]^2)=\left\{f:[0,1+p]^2\to\mathbb{R}|f\ B\text{-continuous on } [0,1+p]^2\right\}$ and $D_b([0,1+p]^2)=\left\{f:[0,1+p]^2\to\mathbb{R}|f\ B\text{-differentiable on } [0,1+p]^2\right\}$.

Let $f\in B_b([0,1+p]^2)$. The function $\omega_{\text{mixed}}(f;\cdot,\cdot):[0,\infty)\times[0,\infty)\to\mathbb{R}$, defined by

$$\omega_{\text{mixed}}(f;\delta_1,\delta_2)=\sup\left\{|\Delta f[(x,y),(s,t)]|:|x-s|\le\delta_1,|y-t|\le\delta_2\right\}\tag{3.1}$$

for any $(\delta_1,\delta_2)\in[0,\infty)\times[0,\infty)$ is called the mixed modulus of smoothness.

For related topics, see [1], [2], [3] and [4].

Let $L:C_b([0,1+p]^2)\to B(\Delta_2)$ be a linear positive operator. The operator $UL:C_b([0,1+p]^2)\to B(\Delta_2)$ defined for any function $f\in C_b([0,1+p]^2)$ and any $(x,y)\in\Delta_2$ by

$$(ULf)(x,y)=\left(L(f(\cdot,y)+f(x,*)-f(\cdot,*))\right)(x,y) \tag{3.2}$$

is called GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and respectively the second variable. The following theorem is proved in the paper [3].

Theorem 3.1. *Let $L : C_b([0,1+p]^2)\to B(\Delta_2)$ be a linear positive operator and $UL : C_b([0,1+p]^2)\to B(\Delta_2)$ the associated GBS operator. Then for any $f\in C_b([0,1+p]^2)$, any $(x,y)\in\Delta_2$ and any $\delta_1,\delta_2>0$, we have*

$$\begin{aligned}&|f(x,y)-(ULf)(x,y)|\le|f(x,y)|\,|1-(Le_{00})(x,y)| \\ &+\Big[(Le_{00})(x,y)+\delta_1^{-1}\sqrt{\left(L(\cdot-x)^2\right)(x,y)}+\delta_2^{-1}\sqrt{\left(L(*-y)^2\right)(x,y)} \\ &+\delta_1^{-1}\delta_2^{-1}\sqrt{\left(L(\cdot-x)^2(*-y)^2\right)(x,y)}\Big]\omega_{\text{mixed}}(f;\delta_1,\delta_2).\end{aligned} \tag{3.3}$$

For B-differentiable functions, we have (see [13]):

Theorem 3.2. *Let $L : C_b([0,1+p]^2)\to B(\Delta_2)$ be a linear positive operator and $UL : C_b([0,1+p]^2)\to B(\Delta_2)$ the associated GBS operator. Then for any $f\in D_b([0,1+p]^2)$ with $D_Bf\in B([0,1+p]^2)$, any $(x,y)\in\Delta_2$ and any $\delta_1,\delta_2>0$, we have*

$$\begin{aligned}&|f(x,y)-(ULf)(x,y)| \\ &\le|f(x,y)||1-(Le_{00})(x,y)|+3\|D_Bf\|_\infty\sqrt{\left(L(\cdot-x)^2(*-y)^2\right)(x,y)} \\ &+\Big[\sqrt{(L(\cdot-x)^2(*-y)^2)(x,y)}+\delta_1^{-1}\sqrt{(L(\cdot-x)^4(*-y)^2)(x,y)} \\ &+\delta_2^{-1}\sqrt{(L(\cdot-x)^2(*-y)^4)(x,y)} \\ &+\delta_1^{-1}\delta_2^{-1}\left(L(\cdot-x)^2(*-y)^2\right)(x,y)\Big]\omega_{\text{mixed}}(D_Bf;\delta_1,\delta_2).\end{aligned} \tag{3.4}$$

Lemma 3.1. *There exists a natural number $m_1\in\mathbb{N}$ such that*

$$(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot-x)^2(*-y)^2)(x,y)\le\frac{1}{4(m+\beta_1)(m+\beta_2)}, \tag{3.5}$$

$$(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot-x)^4(*-y)^2)(x,y)\le\frac{1}{4(m+\beta_1)^2(m+\beta_2)} \tag{3.6}$$

and

$$(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot-x)^2(*-y)^4)(x,y)\le\frac{1}{4(m+\beta_1)(m+\beta_2)^2}, \tag{3.7}$$

for any $m\in\mathbb{N}$, $m\ge m_1$ and any $(x,y)\in\Delta_2$.

Proof. Using the relations $(m+\beta_1)^i(m+\beta_2)^j(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}e_{ij})(x,y) = \sum_{\nu_1=0}^{i}\sum_{\nu_2=0}^{j}\binom{i}{\nu_1}\binom{j}{\nu_2}m^{\nu_1+\nu_2}\alpha_1^{i-\nu_1}\alpha_2^{j-\nu_2}(B_{m+p}e_{\nu_1\nu_2})(x,y)$, for any $i,j\in\mathbb{N}_0$, we get $(m+\beta_1)^2(m+\beta_2)^2(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot-x)^2(*-y)^2)(x,y) = Am^2+Bm+C$, where A,B,C are real numbers depending on $p,x,y,\alpha_1,\beta_1,\alpha_2,\beta_2$ and $A = xy(1-x)(1-y)+2x^2y^2 \le 3/16$. Further on, we have $(m+\beta_1)^4(m+\beta_2)^2(\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(\cdot-x)^2(*-y)^2)(x,y) = Am^3+Bm^2+Cm+D$, where A,B,C,D are real numbers depending on $p,x,y,\alpha_1,\beta_1,\alpha_2,\beta_2$ and $A = 3x(1-x)[xy(1-x)(1-y)+4x^2y^2] \le 15/64$. We used the inequalities $x(1-x)\le 1/4$, for any $x\in[0,1]$ and $xy(1-x)(1-y)\le 1/16$, $xy\le 1/4$, $x^2y^2\le 1/16$, for any $(x,y)\in\Delta_2$. □

Theorem 3.3. *If $f\in C_b([0,1+p]^2)$, then for any $(x,y)\in\Delta_2$ and any $m\in\mathbb{N}$, $m\ge m_1$, the following inequalities*

$$|(U\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}f)(x,y)-f(x,y)| \le \left(1+\delta_1^{-1}\sqrt{\frac{m+p+4\gamma_1}{4(m+\beta_1)^2}}\right. \tag{3.8}$$

$$\left.+\delta_2^{-1}\sqrt{\frac{m+p+4\gamma_2}{4(m+\beta_2)^2}}+\delta_1^{-1}\delta_2^{-1}\frac{1}{2\sqrt{(m+\beta_1)(m+\beta_2)}}\right)\omega_{mixed}(f;\delta_1,\delta_2),$$

for any $\delta_1,\delta_2>0$ and

$$|(U\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}f)(x,y)-f(x,y)| \tag{3.9}$$

$$\le \frac{5}{2}\omega_{mixed}\left(f;\sqrt{\frac{m+p+4\gamma_1}{m^2}},\sqrt{\frac{m+p+4\gamma_2}{m^2}}\right)$$

hold, where

$$(U\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}f)(x,y) = \sum_{\substack{k,j=0\\k+j\le m+p}} p_{m+p,k,j}(x,y)\left(f\left(\frac{k+\alpha_1}{m+\beta_1},y\right)\right.$$

$$\left.+f\left(x,\frac{j+\alpha_2}{m+\beta_2}\right)-f\left(\frac{k+\alpha_1}{m+\beta_1},\frac{j+\alpha_2}{m+\beta_2}\right)\right).$$

Proof. For the first inequality, we apply Theorem 3.1 and Lemma 3.1. The inequality (3.9) is obtained from (3.8) by choosing $\delta_1 = \sqrt{\frac{m+p+4\gamma_1}{m^2}}$ and $\delta_2 = \sqrt{\frac{m+p+4\gamma_2}{m^2}}$. □

Corollary 3.1. *If $f\in C_b([0,1+p]^2)$, then*

$$\lim_{m\to\infty} U\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}f = f \tag{3.10}$$

uniformly on Δ_2.

Proof. It results from the relation (3.9). □

Theorem 3.4. *Let the function* $f \in D_b([0,1+p]^2)$ *with* $D_B f \in B([0,1+p]^2)$*. Then, for any* $(x,y) \in \Delta_2$*, any* $m \in \mathbb{N}$*,* $m \geq m_1$*, we have*

$$|(U\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)} f)(x,y) - f(x,y)| \leq \frac{3}{2\sqrt{m+\beta_1}\sqrt{m+\beta_2}}\|D_B f\|_\infty \tag{3.11}$$
$$+ \frac{1}{2\sqrt{m+\beta_1}\sqrt{m+\beta_2}}\left(1 + \delta_1^{-1}\frac{1}{\sqrt{m+\beta_1}} + \delta_2^{-1}\frac{1}{\sqrt{m+\beta_2}}\right.$$
$$\left. + \delta_1^{-1}\delta_2^{-1}\frac{1}{2\sqrt{m+\beta_1}\sqrt{m+\beta_2}}\right)\omega_{\text{mixed}}(D_B f;\delta_1,\delta_2),$$

for any $\delta_1, \delta_2 > 0$ *and*

$$|(U\tilde{S}_{m,p}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)} f)(x,y) - f(x,y)| \leq \frac{3}{2\sqrt{m+\beta_1}\sqrt{m+\beta_2}}\|D_B f\|_\infty \tag{3.12}$$
$$+ \frac{7}{4\sqrt{m+\beta_1}\sqrt{m+\beta_2}}\omega_{\text{mixed}}\left(D_B f; \frac{1}{\sqrt{m+\beta_1}}, \frac{1}{\sqrt{m+\beta_2}}\right).$$

Proof. It results from Theorem 3.2 and Lemma 3.1. □

Remark 3.1. For $p = 0$ we obtain the results from the paper [11] and for $\beta_1 = \beta_2 = 0$ we obtain the results from the paper [10].

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