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ON A QUADRATIC PENCIL OF DIFFERENTIAL OPERATORS WITH PERIODIC GENERALIZED POTENTIAL

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Abstract: In this paper, we obtain conditions for the absence of spectral gaps in spectrum of a quadratic pencil of Strum-Liouville operarors with periodic generalized potential.

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1. Introduction

The problem we will study in this article is a continuation of problem which was investigated in Manafov at al [6] and [5]. The problem of studying spectral characteristics of the differential equation

$$\ell_\alpha[y] \equiv -y'' + 2\alpha\lambda \sum_{n=-\infty}^{\infty} \delta(x-n)y + q(x)y = \lambda^2 y,\ x \in R \tag{1}$$

arises in many model problems of quantum mechanics. In this formula $q(x)$ is

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a real, non-negative, periodic, piece-wise continuous function ($q(x+1) = q(x)$); $\delta(x)$ is Dirac's function; $\alpha \neq 0$ is a real number and λ is a spectral parameter.

We note that references for the spectrum of the equation (1) and the eigenfunction expansion for $\alpha = 0$ are Titchmarsh [8] and Eastham [2].

First we denote by L_α^λ the consisting operator of differential expression $\ell_\alpha[y]$ in $L_2(R)$. By using the method in Atkinson [1], if we formally integrate the both sides of the equality (1) over the interval $[n-\gamma, n+\gamma]$ and then pass to limit as $\gamma \to 0$, we will get: The operator L_α^λ is equivalent to a differential operator

$$L_0^\lambda : y(x) \to -y''(x) + q(x)y \tag{2}$$

with dense definition domain such that $y(x) \in W_2^2(R\backslash Z) \bigcap W_2^1(R)$ and for $n \in Z$, $y(n) = y(n+0) = y(n-0)$, $\quad y'(n+0) - y'(n-0) = 2\alpha\lambda y(n)$.

Now let us consider following $t-$periodic boundary problem for $0 \leq x \leq 1$ corresponds to the operator (2)

$$y'' + [\lambda^2 - q(x)]y = 0\,, \tag{3}$$

$$y(1) = e^{it}y(0), \quad y'(1) = e^{it}[y'(0) + 2\alpha\lambda y(0)]\,, \tag{4}$$

where t is a real number.

We will denote by $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ the solutions of the equation (3) such that this solutions satisfy the following conditions at $x = 0$

$$\theta(0,\lambda) = \varphi'(0,\lambda) = 1, \quad \theta'(0,\lambda) = \varphi(0,\lambda) = 0. \tag{5}$$

Following results were obtained in Manafov at al [6] and [5] by using the methods in Titchmarsh [8], Eastham [2], Mc Garvey [7] and Guseinov [4].

1. The eigenvalues of the boundary problem (3), (4) are real and zeros of the equation $F(\lambda) \equiv \varphi'(1,\lambda) + \theta(1,\lambda) - 2\alpha\lambda\varphi(1,\lambda)) = 2\cos t$.

2. All eigenvalues $\lambda_k(t)$ $(k = 0, \pm 1, \pm 2, ...)$ of the boundary problem (3), (4) are nonzero real numbers and satisfy the following inequalities

$$... \leq \lambda_{-2}(t) \leq \lambda_{-1}(t) \leq \lambda_0(t) \leq \lambda_1(t) \leq \lambda_2(t)... \tag{6}$$

3. Let $\alpha_{2k}^{\pm}$, $\alpha_{2k+1}^{\pm}$ $(k = 0, \pm 1, \pm 2, ...)$ be eigenvalues of the periodic $(t = 0)$ and anti-periodic $(t = \pi)$ boundary problems (3), (4) respectively. Then the following inequalities are valid

$$...\alpha_{-2}^- \leq \alpha_{-2}^+ < \alpha_{-1}^- \leq \alpha_{-1}^+ < \alpha_0^- \leq \alpha_0^+ < \alpha_1^- \leq \alpha_1^+ < \alpha_2^- \leq \alpha_2^+ < ...$$

4. The spectral intervals of the operator L_α^λ are $(\alpha_{k-1}^+, \alpha_k^-)$ $(k = 0, \pm 1, \pm 2, ...)$ and the gaps are (α_k^-, α_k^+) $(k = 0, \pm 1, \pm 2, ...)$.

5. The eigenfunction expansion of the operator L_α^λ was obtained in Manafov

at al [5]

The main gaol of our paper is to find conditions for the absence of spectral gaps in the spectrum of investigated problem.

2. Constructive Boundary Value Problem

Let us consider the following constructive boundary value problem in the interval $(0,1)$ to arrive our aim.

$$y'' + [\lambda^2 - q(x)]y = 0\,, \tag{7}$$

$$y'(0) + 2\alpha\lambda y(0) = 0, \quad y'(1) = 0\,. \tag{8}$$

From the above results we know the eigenvalues of (3), (4) are real and simple. Now we denote these eigenvalues with following

$$... < \beta_{-3} < \beta_{-2} < \beta_{-1} < \beta_0 = 0 < \beta_1 < \beta_2 < \beta_3 < ...$$

Theorem 1. *There is only one eigenvalue of the problem (7), (8) in the each gaps in spectrum of L_α^λ, i.e. $\beta_k \in (\alpha_k^-, \alpha_k^+)$ $(k = \pm1, \pm2, ...)$.*

Proof. It is easily seen that the eigenvalues of (7), (8) are the roots of the equation $\theta'(1,\lambda) - 2\alpha\lambda\varphi'(1,\lambda) = 0$. So from the equation $\theta'(1,\beta_k) = 2\alpha\beta_k\varphi'(1,\beta_k)$ and Wronskian $W(\theta,\varphi) = 1$ we arrive at

$$\begin{aligned} F(\beta_k) &= \theta(1,\beta_k) + \varphi'(1,\beta_k) - 2\alpha\beta_k\varphi(1,\beta_k) \\ &= \theta(1,\beta_k) + \varphi'(1,\beta_k) - \frac{\theta'(1,\beta_k)}{\varphi'(1,\beta_k)} = \varphi'(1,\beta_k) + \frac{1}{\varphi'(1,\beta_k)} \end{aligned} \tag{9}$$

and

$$F(\beta_k).[Sgn\varphi'(1,\beta_k)] \geq 2. \tag{10}$$

Now taking the derivative to (7) according to λ we obtain

$$\frac{\partial^3 y\,(x,\lambda)}{\partial x^2 \partial\lambda} + \left[\lambda^2 - q(x)\right]\frac{\partial y\,(x,\lambda)}{\partial\lambda} = -2\alpha\lambda y\,(x,\lambda)\,. \tag{11}$$

Let us denote the solutions of (11) by $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ such that the following conditions are satisfied

$$\frac{\partial\theta\,(0,\lambda)}{\partial\lambda} = \frac{\partial\varphi\,(0,\lambda)}{\partial\lambda} = \frac{\partial\theta'\,(0,\lambda)}{\partial\lambda} = \frac{\partial\varphi'\,(0,\lambda)}{\partial\lambda} = 0\,. \tag{12}$$

So the solutions of the non-homogeneous initial value problem (11), (12) will be like following forms

$$\frac{\partial\varphi(x,\lambda)}{\partial\lambda}=2\lambda\int_0^x\{\theta(x,\lambda)\varphi(\xi,\lambda)-\varphi(x,\lambda)\theta(\xi,\lambda)\}\varphi(\xi,\lambda)\,d\xi\,,$$

$$\frac{\partial\theta(x,\lambda)}{\partial\lambda}=2\lambda\int_0^x\{\theta(x,\lambda)\varphi(\xi,\lambda)-\varphi(x,\lambda)\theta(\xi,\lambda)\}\theta(\xi,\lambda)\,d\xi\,,$$

thus

$$\frac{\partial\varphi'(x,\lambda)}{\partial\lambda}=2\lambda\int_0^x\left\{\theta'(x,\lambda)\varphi(\xi,\lambda)-\varphi'(x,\lambda)\theta(\xi,\lambda)\right\}\varphi(\xi,\lambda)\,d\xi\,,\tag{13}$$

$$\frac{\partial\theta'(x,\lambda)}{\partial\lambda}=2\lambda\int_0^x\left\{\theta'(x,\lambda)\varphi(\xi,\lambda)-\varphi'(x,\lambda)\theta(\xi,\lambda)\right\}\theta(\xi,\lambda)\,d\xi\,.\tag{14}$$

Taking $x=1$, $\lambda=\beta_k$ in (13), (14) and from the equality $\theta'(1,\beta_k)=2\alpha\beta_k\varphi'(1,\beta_k)$ we obtain

$$\begin{aligned}&\varphi'(1,\beta_k)\left[\tfrac{\partial}{\partial\lambda}\left(2\alpha\lambda\varphi'(1,\lambda)-\theta'(1,\lambda)\right)\right]_{\lambda=\beta_k}\\&\quad=2\varphi'^2(1,\beta_k)\left[\beta_k\int_0^1\{\theta(\xi,\beta_k)-2\alpha\beta_k\varphi(\xi,\beta_k)\}^2\,d\xi+2\alpha\right].\end{aligned}\tag{15}$$

Without loss of generality, from (15), we arrive at

$$\varphi'(1,\beta_k)[\frac{\partial}{\partial\lambda}\left(2\alpha\lambda\varphi'(1,\lambda)-\theta'(1,\lambda)\right)]_{\lambda=\beta_k}>0,\quad\text{if}\quad\beta_k>0\quad(\text{i.e.}\,k\geq1)\,,$$

$$\varphi'(1,\beta_k)[\frac{\partial}{\partial\lambda}\left(2\alpha\lambda\varphi'(1,\lambda)-\theta'(1,\lambda)\right)]_{\lambda=\beta_k}<0,\quad\text{if}\quad\beta_k<0\quad(\text{i.e.}\,k\leq-1)\,.$$

Thus

$$Sgn\varphi'(1,\beta_k)=Sgn[\frac{\partial}{\partial\lambda}\left(2\alpha\lambda\varphi'(1,\lambda)-\theta'(1,\lambda)\right)]_{\lambda=\beta_k},\quad\text{if}\quad k\geq1,\tag{16}$$

$$Sgn\varphi'(1,\beta_k)=-Sgn[\frac{\partial}{\partial\lambda}\left(2\alpha\lambda\varphi'(1,\lambda)-\theta'(1,\lambda)\right)]_{\lambda=\beta_k},\quad\text{if}\quad k\leq-1\,.\tag{17}$$

Since the zeros of the equation $\theta'(1,\lambda)=2\alpha\lambda\varphi'(1,\lambda)$ are simple, the following inequalities are valid

$$[\frac{\partial}{\partial\lambda}\left(2\alpha\lambda\varphi'(1,\lambda)-\theta'(1,\lambda)\right)]_{\lambda=\beta_1}<0,$$

$$[\frac{\partial}{\partial\lambda}\left(2\alpha\lambda\varphi'(1,\lambda)-\theta'(1,\lambda)\right)]_{\lambda=\beta_2}>0,\ldots,$$

$$[\frac{\partial}{\partial\lambda}\left(2\alpha\lambda\varphi'(1,\lambda)-\theta'(1,\lambda)\right)]_{\lambda=\beta_{-1}}<0,$$

$$[\frac{\partial}{\partial\lambda}\left(2\alpha\lambda\varphi'\left(1,\lambda\right)-\theta'\left(1,\lambda\right)\right)]_{\lambda=\beta_{-}2}>0,\dots.$$

From (16) and (17) we obtain

$$sgn\varphi'(1,\beta_k)=(-1)^k \quad (k=\pm1,\pm2,...)\,,$$

from (10) we arrive

$$F(\beta_k).(-1)^k\geq 2 \quad (k=\pm1,\pm2,...)\,,$$

i.e.

$$\begin{gathered} F(\beta_1)\leq -2,\ F(\beta_2)\geq 2,\ F(\beta_3)\leq -2,\dots, \\ F(\beta_{-1})\leq -2,\ F(\beta_{-2)}\geq 2,\ F(\beta_{-3})\leq -2,\dots. \end{gathered} \tag{18}$$

This inequalities show that two different sequences of the eigenvalues of (7), (8) lie in different gaps. In other words there is not much more one β_k in each gaps. Now finally, we will show the existence of eigenvalues in each gaps.

Since the function $F(\lambda)$ is a continuous function and from the inequality $F(0)>2$ and (18), the values of the function $F(\lambda)$ can be $+2$ and -2 only once in the following intervals.

$$[0,\beta_1],\ [\beta_1,\beta_2],\ [\beta_2,\beta_3],...[\beta_{-1},0],\ [\beta_{-2},\beta_{-1}],\ [\beta_{-3},\beta_{-2}],\dots. \tag{19}$$

Let us denote by $\tilde{\alpha}_k^+\in[\beta_k,\beta_{k+1}]$ ($\tilde{\alpha}_k^-\in[\beta_{k-1},\beta_k]$) the near points of β_k which roots of the equation $F(\lambda)=2(-1)^k$, where $\beta_0\equiv 0$. Then

$$..\tilde{\alpha}_{-1}^-\leq\beta_{-1}\leq\tilde{\alpha}_{-1}^+<\tilde{\alpha}_0^-<0<\tilde{\alpha}_0^+<\tilde{\alpha}_1^-\leq\beta_1\leq\tilde{\alpha}_1^+<\tilde{\alpha}_2^-\leq\beta_2\leq\tilde{\alpha}_2^+\dots. \tag{20}$$

Furthermore it is seen that the following asymptotic formulas are correct for the eigenvalues $\{\beta_k\}$ of periodic ($t=0$) and anti-periodic ($t=\pi$) problems (7), (8) and the eigenvalues $\{\alpha_k\}$ of the problem (3), (4) as $|k|\to\infty$

$$\begin{gathered} \beta_k^{\pm}=2\pi k-\arctan 2\alpha+\frac{a_0}{8\pi k}+\frac{\delta^{\pm}}{k}\,, \\ \alpha_k^{\pm}=2\pi k+\arctan\alpha+\frac{a_0}{8\pi k}+\frac{\varepsilon^{\pm}}{k}\,, \end{gathered} \tag{21}$$

where

$$\sum_k\left\{\left|\delta_k^{\pm}\right|^2+\left|\varepsilon_k^{\pm}\right|^2\right\}<\infty \tag{22}$$

and

$$a_0=2\int_0^1 q(y)dy\,. \tag{23}$$

From (20) and (21) we arrive that the sequence $\{\alpha_k^{\pm}\}$ contains all eigenvalues of periodic and anti-periodic problems. $\square$

3. Regularized Trace Formulas

From the asymptotic formulas (21) we obtain following converge series

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\beta_k+\beta_{-k}+2\arctan 2\alpha\right),\\ &\sum_{k=1}^{\infty}\left(\beta_k^2+\beta_{-k}^2-8\pi^2k^2-2\arctan^2 2\alpha-\tfrac{a_0}{\pi}\right).\end{aligned} \tag{24}$$

It is seen that the following formulas are valid by using the method in Guseinov at al [4]

$$\sum_{k=1}^{\infty}\left(\beta_k+\beta_{-k}+2\arctan 2\alpha\right)=-\frac{q(0)+q(1)}{2}+\int_0^1 q(x)dx\,, \tag{25}$$

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\beta_k^2+\beta_{-k}^2-8\pi^2k^2-2\arctan^2 2\alpha-\tfrac{a_0}{\pi}\right)\\ &\qquad=-\tfrac{q^2(0)+q^2(1)}{2}+\int_0^1 q^2(x)dx+\left(\int_0^1 q(x)dx\right)^2.\end{aligned} \tag{26}$$

4. Conditions for the Absence of Spectral Gaps

We assume that τ is real parameter. Let us consider the equation (1) and following pencil of operator

$$-y''+2\alpha\lambda\sum_{n=-\infty}^{\infty}\delta(x+\tau-n)y+q(x+\tau)y=\lambda y\,. \tag{27}$$

For each τ the pencil of operator generated by the equation (27) in $L_2(R)$ is equivalent to a differential operator $-y''+q(x+\tau)y$ with dense definition domain such that $y(x)\in W_2^2(R\setminus Z)\cap W_2^1(R)$ and for $n\in Z$, $y(n)=y(n+0)=y(n-0)$, $y'(n+0)-y'(n-0)=2\alpha\lambda y(n)$. Here the function $F(\lambda,\tau)\equiv\varphi'(1,\lambda;\tau)+\theta(1,\lambda;\tau)-2\alpha\lambda\varphi(1,\lambda;\tau)$ does not dependent on the parameter τ and equal to $F(\lambda)$. Here $\varphi(x,\lambda;\tau)$ and $\theta(x,\lambda;\tau)$ are the solutions of the following equation and conditions

$$-y''+q(x+\tau)y=\lambda^2 y\,, \tag{28}$$

$$\varphi(0,\lambda;\tau)=\theta'(0,\lambda;\tau)=0,\quad \varphi'(0,\lambda;\tau)=\theta(0,\lambda;\tau)=1\,. \tag{29}$$

By θ' we designate the derivative of θ with respect to x. So spectrum of periodic and anti-periodic boundary problem generated by the equation (27) for $0\le x\le 1$ does not depend on the parameter τ. But the eigenvalues $\beta_k(\tau)$ of the problem

(28), (8) will be a function of τ and periodic with period 1. Let us write trace formulas (25), (26) for the problem (28), (8)

$$\sum_{k=1}^{\infty}(\beta_k(\tau)+\beta_{-k}(\tau)+2\arctan 2\alpha)=-q(\tau)+\int_0^1 q(x)dx\,, \tag{30}$$

$$\sum_{k=1}^{\infty}(\beta_k^2(\tau)+\beta_{-k}^2(\tau)-8\pi^2k^2-2\arctan^2 2\alpha-\tfrac{a_0}{\pi}) \\ =q^2(\tau)+\tfrac{1}{2}\int_0^1 q^2(x)dx+\left(\int_0^1 q(x)dx\right)^2. \tag{31}$$

From Theorem 1 for all τ

$$\beta_k(\tau)\in(\alpha_k^-,\alpha_k^+)\quad (k=\pm1,\pm2,...)\,. \tag{32}$$

We note here that the independence of function $\beta_k(\tau)$ from the parameter τ was demonstrated in Guseinov at al [3].

If the gap (α_0^-,α_0^+) contain the point $\lambda=0$ then this gap will be a non-singular gap. If $\alpha_0^-=\alpha_0^+$ then this gap said trivial gap and remain non-singular gaps (α_k^-,α_k^+) are said non-trivial gaps.

From (32), if all non-trivial gaps do not absent in the spectrum of pencil operator L_α^λ, i.e. $\alpha_k^-=\alpha_k^+$, and $k=\pm1,\pm2,...$, then all functions $\beta_k(\tau)$ will be constant. So we arrive that $q(\tau)$ given in formulas (30), (31) is a constant function. It is clear that the inverse is true. So we obtain the following theorem.

Theorem 2. *A necessary and sufficient condition for the absence of gaps in the spectrum of pencil of operator L_α^λ is that $\alpha\equiv0$ and $q(x)$ is a constant function.*

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