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COMPLEXITY METRIC AND STRUCTURAL MEASURE ON THE CLASS OF NON DETERMINISTIC MATRICES

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Abstract: This paper deals with the definitions of product of non-deterministic finite automaton, non-deterministic digraph, non-deterministic matrix, complexity metric and structural measure on the class of non-deterministic matrices. These definitions are used to establish the relation between structural measure and the complexity metric on the class of non-deterministic matrices.

AMS Subject Classification: 05C
Key Words: non-deterministic digraph, non-deterministic matrix, complexity metric

1. Introduction

A graph based structural measure and complexity metric is established in [3]. Product of finite automata, deterministic digraph, deterministic matrix, complexity metric and structural measure on the class of deterministic matrices are developed in [6]. In this paper, we extend the theory to non-deterministic digraphs and non-deterministic matrices.

In Section 2, we present preliminaries on product of digraphs of automata.

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For the general theory on automata one can refer [2] and the references cited therein. In Section 3, we present the conversion of non-deterministic digraphs into deterministic digraphs, the product of non-deterministic digraphs, properties of product graphs. In Section 4, we establish the relation between the non-deterministic matrix and the corresponding deterministic matrix with the relation between the complexity metric and structural measure on the class of non-deterministic matrices.

Throughout this paper we use the following notations.

(i) $(a_i)_1^m = (a_1, a_2, \ldots, a_m)$.

(ii) $\prod_1^m A_i = \{(a_1, a_2, \ldots, a_m) : a_i \in A_i, i = 1, 2, \ldots, m\}$ for sets A_i.

(iii) $\prod_1^m n_i = n_1 \times n_2 \times \ldots \times n_m$ for non-negative integers n_i.

(iv) $|X|$ denotes the number of elements in the set X.

(v) DG = Deterministic digraph.

(vi) NG = Non-deterministic digraph.

(vii) 2^Q = Set of all subsets of Q.

2. Preliminaries

In this section, we present some basic definitions and results on deterministic and non-deterministic finite automata.

Definition 2.1. (see [2]) A Deterministic Finite Automata (DFA) is a 5 - tuple $M = (Q, \sum, \delta, q_0, F)$, where Q is a finite set of states, $\sum$ is a finite set of input alphabets, $q_0 \in Q$ is the initial state, $F \subset Q$ is the set of final states and $\delta : Q \times \sum \to Q$ is a transition function. If there exists a function $\delta' : Q \times \sum \to 2^Q$, then $M = (Q, \sum, \delta', q_0, F)$ is a non-deterministic Finite Automata (NFA) and δ' is called the transition function.

A directed graph, called a transition diagram is associated with a Finite Automaton (FA) as follows. The vertices of the graph correspond to the states of the FA. If $\delta(q, a) = p$, then there is an arrow labeled a from the vertex q to the vertex p in the transition diagram. The FA accepts a string x if the sequence of transitions corresponding to the symbols of x leads from the start to an accepting finial state.

Definition 2.2. (see [6]) For $i = 1, 2, 3, \ldots m$, let $M_i = (Q_i, \sum_i, \delta_i, q_{0i}, F_i)$

be any m deterministic finite automata and $p_i = \delta_i(q_i, a_i)$. Product of deterministic finite automata (PDFA) is a 5 - tuple $M = (Q, \sum, \delta, q_0, F)$, where $Q = \prod_1^m Q_i$, $\sum = \prod_1^m \sum_i$, $q_0 = (q_{0i})_1^m$, $F = \prod_1^m F_i$ and $\delta : Q \times \sum \to Q$ is defined as $\delta((q_i)_1^m, (a_i)_1^m) = (p_i)_1^m$. By identifying $(x_i)_1^m (y_i)_1^m = (x_i y_i)^m$, the language accepted by M is defined as $L(M) = \{(x_i)_1^m \in \sum^*; x_i \in \sum_i$ and $\delta_i(q_{0i}, x_i) \in F_i\}$, where $\sum^*$ is the set of all strings formed by finite elements of $\sum$. The transition digraph of M is a digraph containing $\prod_1^m |Q_i|$ vertices with vertex label set $\prod_1^m Q_i$ and for each $\delta((q_i)_1^m, (a_i)_1^m) = (p_i)_1^m$ there is an arrow from the vertex $(q_i)_1^m$ to $(p_i)_1^m$ with edge label $(a_i)_1^m$.

Definition 2.3. Consider a digraph $G = (V, E)$. If all the vertices and edges of the digraph G are labeled in such a way that no two vertices have same label (many edges can have same label), then the digraph G is called a labeled digraph. Denote the set of all labels of vertices of G as Q and edges as $\sum$. If there exists a transition function $\delta : Q \times \sum \to Q$ in G, then the labeled digraph G is called a deterministic digraph (DG) and is denoted as $(Q, \sum, \delta)$. If there exists a function $\delta' : Q \times \sum \to 2^Q$ in G, then the labeled digraph G is called non-deterministic digraph (NG) and is denoted by $(Q, \sum, \delta')$.

Definition 2.4. (see [6]) Let $(Q_i, \sum_i, \delta_i), i = 1, 2, 3, \dots m$ be any m deterministic digraphs as defined in Definition 2.3. Define $Q = \prod_1^m Q_i, \sum = \prod_1^m \sum_i$ and define a function $\delta : Q \times \sum \to Q$ as $\delta((q_i)_1^m, (a_i)_1^m) = (p_i)_1^m$, where $p_i = \delta_i(q_i, a_i)$ for all $q_i \in Q_i$ and $a_i \in \sum_i$. The deterministic digraph $G = (Q, \sum, \delta)$ is called product deterministic digraph (PDG) of the given deterministic digraphs.

Definition 2.5. (see [6]) Consider a deterministic digraph DG $= (Q, \sum, \delta)$. Rename the elements of Q as $1, 2, \dots, |Q|$. If a_{ij} denotes the number of arrows from i to j in DG, then the matrix A(DG) $= (a_{ij})$ is called the adjacency matrix of DG. A square matrix M is said to be a deterministic matrix if it is the adjacent matrix of some DG. ie; M = A(DG).

Lemma 2.6. (see [6]) *Let $A(DG) = (a_{ij})$ be the deterministic matrix of $DG = (Q, \sum, \delta)$. k_{ij} is the (i, j)-th entry of the matrix $A^k(DG)$ if and only if k_{ij} = number of directed paths of length k from i to j.*

Theorem 2.7. (see [6]) *A square matrix A of non-negative integers is a deterministic matrix if and only if all row sums or all column sums of the matrix A are equal.*

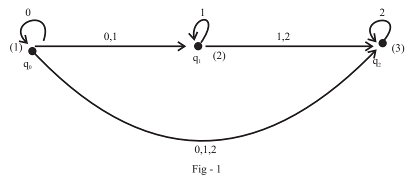

Figure 1:

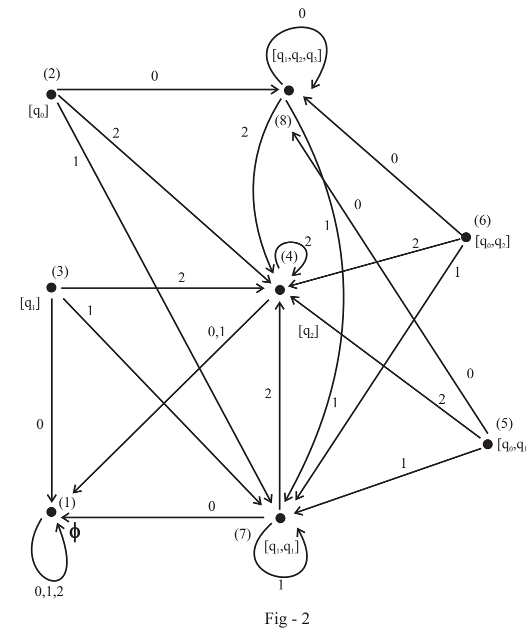

Figure 2:

3. Deterministic Matrices of Non-Deterministic Digraphs

In this section, by defining the *deterministic digraphs*, corresponding to the *non-deterministic digraphs*, we establish some relations between deterministic and non-deterministic matrices.

Definition 3.1. Let NG $= (Q, \sum, \delta')$ be a non-deterministic digraph defined as in Definition 2.3. Construct a deterministic digraph DG $= (2^Q, \sum, \delta)$ as follows. The set of labels of vertices of DG is the set of all subsets of Q and set of labels of edges of DG is $\sum$. A vertex of DG will be denoted by $[q_1 q_2 \dots q_i]$ where $q_1, q_2, \dots, q_i$ are in Q. Observe that $[q_1 q_2 \dots q_i]$ is a single vertex of DG. We define the transition function $\delta : 2^Q \times \sum \to 2^Q$ by $\delta\left([q_1 q_2 \dots q_i], a\right) = [p_1 p_2 \dots p_r]$ if and only if $\delta'\left(\{q_1, q_2, ...q_i\}, a\right) = \{p_1, p_2, \dots, p_r\}$ where $\delta'\left(\{q_1, q_2, ...q_i\}, a\right) = \bigcup_{j=1}^{i} \delta'(q_j, a)$ and $\delta(\phi, a) = \phi$ for the empty set ϕ. By relabeling the vertices of DG as $1, 2, 3, \dots, 2^{|Q|}$ the adjacency matrix of the DG is called a deterministic matrix of the corresponding NG.

Remark 3.2. Every DG is a NG but every NG need not be a DG.

Example 3.3. Consider the non-deterministic digraph NG $= (Q, \sum, \delta')$

where $Q = \{q_0, q_1, q_2\}, \sum = \{0, 1, 2\}$ and $\delta' : Q \times \sum \to 2^Q$ given by

δ'	0	1	2
q_0	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_2\}$
q_1	ϕ	$\{q_1, q_2\}$	$\{q_2\}$
q_2	ϕ	ϕ	$\{q_2\}$

Table 1:

Figure 1 is the transition diagram of the NG given in Table 1.

The adjacency matrix of Figure 1 is $A(NG) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ which is a non-deterministic matrix.

By Definition 3.1, the deterministic *digraph* of NG given in Figure 1 is DG $= (2^Q, \sum, \delta)$ where $\delta : 2^Q \times \sum \to 2^Q$ given by

δ	0	1	2
ϕ	ϕ	ϕ	ϕ
$[q_0]$	$[q_0q_1q_2]$	$[q_1q_2]$	$[q_2]$
$[q_1]$	ϕ	$[q_1q_2]$	$[q_2]$
$[q_2]$	ϕ	ϕ	$[q_2]$
$[q_0q_1]$	$[q_0q_1q_2]$	$[q_1q_2]$	$[q_2]$
$[q_0q_2]$	$[q_0q_1q_2]$	$[q_1q_2]$	$[q_2]$
$[q_1q_2]$	ϕ	$[q_1q_2]$	$[q_2]$
$[q_0q_1q_2]$	$[q_0q_1q_2]$	$[q_1q_2]$	$[q_2]$.

Table 2:

Figure 2 is the 3-regular DG of the Figure 1. By Definition 3.1, the adjacency matrix of Figure 2 is the deterministic matrix which is given below whose row sums are equal.

$$A(DG) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Definition 3.4. Let $(2^{Q_i}, \sum_i, \delta_i), i = 1, 2, \ldots, m$ be m deterministic digraphs corresponding to the non-deterministic digraphs $(Q_i, \sum_i, \delta_i')$ respectively. Take $Q = \prod_1^m 2^{Q_i}, \sum = \prod_1^m \sum_i$ and define a function $\delta : Q \times \sum \to Q$ as

$$\delta\left(\left(\left[q_{i_1}q_{i_2}...q_{i_j}\right]\right)_{i=1}^m, (a_i)_{i=1}^m\right) = \left([p_{i_1}p_{i_2}...p_{i_r}]\right)_{i=1}^m ,$$

where $\delta_i(\phi_i, a_i) = \phi_i$ for empty set ϕ_i and $\delta_i([q_{i_1}q_{i_2}\dots q_{i_j}], a_i) = [p_{i_1}p_{i_2}\dots p_{i_r}]$, $i = 1, 2, \dots, m$. The deterministic digraph denoted by $PG = (Q, \sum, \delta)$ is the *product graph* of the given *non-deterministic digraphs*. By relabeling the vertices of Q as $1, 2, \dots, |Q|$ the adjacency matrix of PG is called *product matrix*.

Example 3.5. Consider the non-deterministic digraphs $NG_1 = (Q_1, \sum_1, \delta_1')$ and $NG_2 = (Q_2, \sum_2, \delta_2')$ where $Q_1 = \{p_1, p_2\}$, $Q_2 = \{q_1, q_2\}, \sum_1 = \sum_2 = \{0, 1\}$ and $\delta_1' : Q_1 \times \sum_1 \to 2^{Q_1}$, $\delta_2' : Q_2 \times \sum_2 \to 2^{Q_2}$ are given as in Table 3, Table 4 respectively.

δ_1'	0	1
p_1	$\{p_2\}$	ϕ_1
p_2	ϕ_1	$\{p_1\}$

Table 3:

δ_2'	0	1
q_1	$\{q_2\}$	$\{q_2\}$
q_2	$\{q_1, q_2\}$	ϕ_2

Table 4:

(Here we denote $\phi_1 = \phi_2 = \phi =$ empty set). Figure 3 and Figure 4 are the non-deterministic digraphs representing NG_1 and NG_2 respectively.

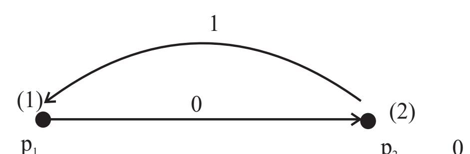

Fig - 3 (N G1)

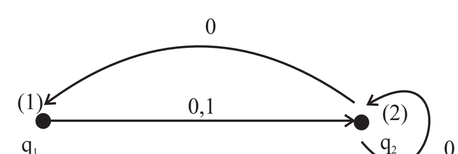

Fig - 4 (N G2)

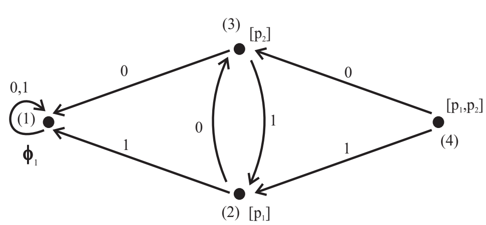

Fig - 5 (D G1)

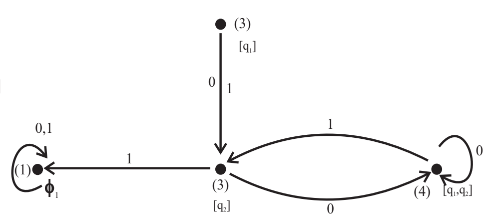

Fig - 6 (DG2)

By Definition 3.1, we obtain $DG_1 = (2^{Q_1}, \sum_1, \delta_1)$ and $DG_2 = (2^{Q_2}, \sum_2, \delta_2)$ where $\delta_1 : 2^{Q_1} \times \sum_1 \to 2^{Q_1}$ and $\delta_2 : 2^{Q_2} \times \sum_2 \to 2^{Q_2}$ are given as in Table 5 and Table 6 respectively.

δ_1	0	1
ϕ_1	ϕ_1	ϕ_1
$[p_1]$	$[p_2]$	ϕ_1
$[p_2]$	ϕ_1	$[p_1]$
$[p_1p_2]$	$[p_2]$	$[p_1]$

Table 5:

δ_2	0	1
ϕ_2	ϕ_2	ϕ_2
$[q_1]$	$[q_2]$	$[q_2]$
$[q_2]$	$[q_1q_2]$	ϕ_2
$[q_1q_2]$	$[q_1q_2]$	$[q_2]$

Table 6:

From Table 5 and Table 6, we can draw DG_1 (Figure 5) and DG_2 (Figure 6). Using Definition 3.4, we construct the product graph $PG = (Q, \sum, \delta)$ of DG_1 and DG_2 where

$$Q = \{(\phi_1, \phi_2), (\phi_1, [q_1]), (\phi_1, [q_2]), (\phi_1, [q_1q_2]), ([p_1], \phi_2), ([p_1], [q_1]), ([p_1], [q_2]), ([p_1], [q_1q_2]), ([p_2], \phi_2), ([p_2], [q_1]), ([p_2], [q_2]), ([p_2], [q_2q_2]), ([p_1p_2], \phi_2), ([p_1p_2], [q_1]), ([p_1p_2], [q_2]), ([p_1p_2], [q_1q_2])\},$$

$$\sum = \{(0,0), (0,1), (1,0), (1,1)\},$$

and the transition function $\delta : Q \times \sum \to Q$ is given as in Table 7.

	δ	$a_1(0,0)$	$a_2(0,1)$	$a_3(1,0)$	$a_4(1,1)$
1	(ϕ_1, ϕ_2)	(ϕ_1, ϕ_2)	(ϕ_1, ϕ_2)	(ϕ_1, ϕ_2)	(ϕ_1, ϕ_2)
2	$(\phi_1, [q_1])$	$(\phi_1, [q_2])$	$(\phi_1, [q_2])$	$(\phi_1, [q_2])$	$(\phi_1, [q_2])$
3	$(\phi_1, [q_2])$	$(\phi_1, [q_1q_2]$	(ϕ_1, ϕ_2)	$(\phi_1, [q_1q_2])$	(ϕ_1, ϕ_2)
4	$(\phi_1, [q_1q_2])$	$(\phi_1, [q_1q_2])$	$(\phi_1, [q_2])$	$(\phi_1, [q_1q_2])$	$(\phi_1, [q_2])$
5	$([p_1], \phi_2)$	$([p_2], \phi_2)$	$([p_2], \phi_2)$	(ϕ_1, ϕ_2)	(ϕ_1, ϕ_2)
6	$([p_1], [q_1])$	$([p_2], [q_2])$	$([p_2], [q_2])$	$(\phi_1, [q_2])$	$(\phi_1, [q_2])$
7	$([p_1], [q_2])$	$([p_2], [q_1q_2])$	$([p_2], \phi_2)$	$(\phi_1, [q_1q_2])$	(ϕ_1, ϕ_2)
8	$([p_1], [q_1q_2])$	$([p_2], [q_1q_2])$	$([p_2], [q_2])$	$(\phi_1, [q_1q_2])$	$(\phi_1, [q_2])$
9	$([p_2], \phi_2)$	(ϕ_1, ϕ_2)	(ϕ_1, ϕ_2)	$([p_1]), \phi_2$	$([p_1], \phi_2)$
10	$([p_2], [q_1])$	$(\phi_1, [q_2])$	$(\phi_1, [q_2])$	$([p_1], [q_2])$	$([p1], [q_2])$
11	$([p_2], [q_2])$	$(\phi_1, [q_1q_2])$	(ϕ_1, ϕ_2)	$([p_1], [q_1q_2])$	$([p_1], \phi_2)$
12	$([p_2], [q_1q_2])$	$(\phi_1, [q_1q_2])$	$(\phi_1, [q_2])$	$([p_1], [q_1q_2])$	$([p_1], [q_2])$
13	$([p_1p_2], \phi_1)$	$([p_2], \phi_2)$	$([p_2], \phi_2)$	$([p_1], \phi_2)$	$([p_1], \phi_2)$
14	$([p_1p_2], [q_1])$	$([p_2], [q_2])$	$([p_2], [q_2])$	$([p_1], [q_2])$	$([p_1], [q_2])$
15	$([p_1p_2], [q_2])$	$([p_2], [q_1q_2])$	$([p_2], \phi_2)$	$([p_1], [q_1q_2])$	$([p_1], \phi_2)$
16	$([p_1p_2], [q_1q_2])$	$([p_2], [q_1q_2])$	$([p_2], [q_2])$	$([p_1], [q_1q_2])$	$([p_1], [q_2])$

Table 7:

Table 7 generates the product graph (deterministic graph) of NG_1 and NG_2 (or DG_1 and DG_2), which is presented in Figure 7.

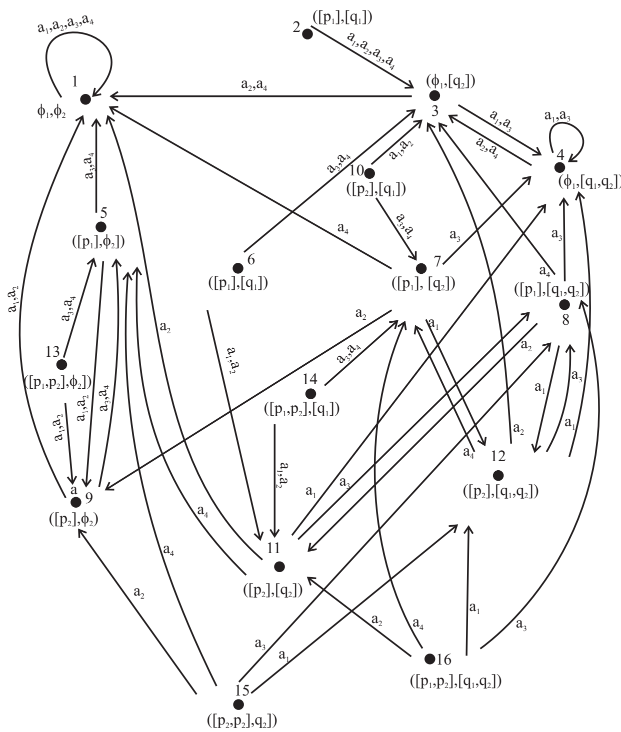

Fig - 7

Figure 7:

Following are the adjacency matrices of the NG's given in Figures 3, 4, 5, 6 and 7, respectively.

$$A(NG_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A(NG_2) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

$$A(DG_1) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad A(DG_2) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$A(PG) = \begin{bmatrix}
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

If row sums of $A(DG_1), A(DG_1), \ldots, A(DG_n)$ are $s_1, s_2, \ldots, s_n$, then the row sum of $A(PG)$ is $\sum\limits_{i=1}^{n} s_i$, where PG is the product graph of $DG_1, DG_2, \ldots,$ DG_n.

Theorem 3.6. *If G is a k-regular directed graph with respect to outgoing degree if and only if G is a deterministic digraph $(Q, \sum, \delta)$, where $|\sum| = k$.*

Proof. The proof follows by taking the vertex set as Q, edge labels by k symbols from $\sum = \{a_1, a_2, \ldots, a_k\}$ and a_i is an arrow from a vertex p to q if and only if $\delta(p, a_i) = q$ and from Definition 2.3. □

Theorem 3.7. *For every non-deterministic digraph $NG = (Q, \sum, \delta')$, there exists a deterministic digraph $DG = (2^Q, \sum, \delta)$ with $2^{|Q|}$ vertices and $2^{|Q|}|\sum|$ directed edges.*

Proof. The proof follows from Definition 3.1 and Theorem 3.6. □

Theorem 3.8. *Let $(Q, \sum, \delta')$ be any NG. If $\sum'$ is any set containing $\sum$, then there exists a $DG = (2^Q, \sum', \delta)$ with $2^{|Q|}$ vertices and $|\sum'|\, 2^{|Q|}$ directed edges.*

Proof. By defining $\delta'(q, b) = \phi$ for all $(q, b) \in Q \times (\sum' - \sum)$, the digraph $(Q, \sum', \delta')$ becomes a NG. Now the proof follows from Theorem 3.7. □

Remark 3.9. Every DG is a regular graph with respect to outgoing degree but the converse need not be true. NG_1 and NG_2 given in Figure 3 and Figure 4 are 1-regular and 2-regular graphs, respectively, but they are not deterministic digraphs since $\delta_1 : Q_1 \times \sum_1 \to Q_1$ and $\delta_2 : Q_2 \times \sum_2 \to Q_2$ do not exist.

Theorem 3.10. *If $(Q_i, \sum_i, \delta_i')$, $i = 1, 2, \ldots, m$ are non-deterministic digraphs, then there exists a product digraph $PG = (Q, \sum, \delta)$ with $\prod_1^m 2^{|Q_i|}$ vertices and $\prod_1^m |\sum_i| \prod_1^m 2^{|Q_i|}$ directed edges.*

Proof. By Theorem 3.7, there exist deterministic digraphs $(2^{Q_i}, \sum_i, \delta_i)$, $i = 1, 2, \ldots, m$ and each $DG_i = (2^{Q_i}, \sum_i, \delta_i)$ has $2^{|Q_i|}$ vertices and $|\sum_i| \; 2^{|Q_i|}$ directed edges. Now the proof follows from Definition 3.4 (see Example 3.5). □

Definition 3.11. A square matrix of non-negative integers is said to be a non-deterministic matrix if it is an adjacency matrix of some non-deterministic digraph $(Q, \sum, \delta')$.

Theorem 3.12. *Every square matrix of non-negative integers is a non-deterministic matrix.*

Proof. Let $A = (a_{ij})_{m\times m}$ be the given square matrix of nonnegative integers of order m. By taking

$$n = \max_{1\le i\le m} \left\{ \sum_{j=1}^{m} a_{ij} \right\}, \tag{1}$$

$Q = \{1, 2, \ldots, m\}$ and $\sum$ is any set containing n different labels $\{a_1, a_2, \ldots, a_n\}$, we shall construct a non-deterministic digraph $NG = (Q, \sum, \delta')$ whose adjacency matrix is A. Take Q as the vertex set of NG and to each pair $(i, j) \in Q \times Q$, select $S_{ij} \subseteq \sum$ containing a_{ij} labels satisfying the following conditions.

$$S_{i0} = \phi, S_{ij} \subseteq \sum - \left(\bigcup_{k=0}^{j-1} S_{ik} \right), |S_{ij}| = a_{ij} \text{ for } i, j = 1, 2, ..., m. \tag{2}$$

From (1) and (2), $\bigcup_{j=1}^{m} S_{ij} \subseteq \sum$ and $\bigcup_{j=1}^{m} S_{ij} = \sum$ for some i, $1 \le i \le m$. Now define $\delta' : Q \times \sum \to 2^Q$ by

$$\delta'(i, a_k) = \begin{cases} \{j\} & \text{if } a_k \in S_{ij}\,, \\ \phi & \text{otherwise}\,, \end{cases} \tag{3}$$

where ϕ is the empty set. Since $|S_{ij}| = a_{ij}$, the function δ' generates a_{ij} arrows from i to j labeled by the elements of S_{ij}. Now the proof follows by taking set of edges of the non-deterministic digraph NG as the collection of all arrows from i to j generated by (3) for all $i, j \in Q$.

The following example illustrates Theorem 3.12. □

Example 3.13. Consider the following square matrix of nonnegative integers.

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 5 & 3 & 0 \\ 6 & 1 & 1 & 2 \\ 7 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\max_{1 \leq i \leq 4} \left\{ \sum_{j=1}^{4} a_{ij} \right\} = \max\{9, 8, 10, 7\} = 10$, as in Theorem 3.12, take $\sum = \{a_1, a_2, \ldots, a_{10}\}$ as the set of edge labels and $Q = \{1, 2, 3, 4\}$ as the set of vertex labels of the required non-deterministic digraph $NG = (Q, \sum, \delta')$.

From (2) and (3), if we take $S_{11} = \{a_1\}, S_{12} = \{a_2, a_3\}, S_{13} = \{a_4, a_5, a_6\}$, $S_{14} = \{a_7, a_8, a_9\}$, $S_{21} = \{\phi\}, S_{22} = \{a_1, a_3, a_5, a_7, a_9\}, S_{23} = \{a_2, a_4, a_6\}, S_{24} = \{\phi\}$, $S_{31} = \{a_1, a_2, a_3, a_4, a_5, a_6\}, S_{32} = \{a_{10}\}, S_{33} = \{a_9\}, S_{34} = \{a_2, a_8\}$, $S_{41} = \{a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\}, S_{42} = S_{43} = S_{44} = \phi$, then $\delta' : Q \times \sum \to 2^Q$ can be taken as in Table 8.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
1	(1)	(2)	(2)	(3)	(3)	(3)	(4)	(4)	(4)	ϕ
2	(2)	(3)	(2)	(3)	(2)	(3)	(2)	ϕ	(2)	ϕ
3	(1)	(1)	(1)	(1)	(1)	(1)	(4)	(4)	(3)	(2)
4	ϕ	ϕ	ϕ	(1)	(1)	(1)	(1)	(1)	(1)	(1)

Table 8:

Figure 8 is the NG of the matrix A, and hence A is a non-deterministic matrix.

Theorem 3.14. *Let k be any positive integer. If $A = (a_{ij})_{m \times m}$ is any non-deterministic matrix of order m with maximum row sum n, then there exists deterministic matrix $B = (b_{ij})_{2^m \times 2^m}$ of order 2^m whose row sums are equal to n, i.e. $RS(B) = n$ and also B^k is a deterministic matrix whose row sum $= n^k$.*

Proof. By Theorem 3.12, there exists a non-deterministic digraph $NG = (Q, \sum, \delta')$ where $Q = \{1, 2, 3, \ldots, m\}$, $\sum$ contains n symbols and $\delta' : Q \times \sum \to 2^Q$ whose adjacency matrix is A. By Theorem 3.7, there exists a deterministic digraph $DG = (2^Q, \sum, \delta)$ with $2^{|Q|}$ vertices and $2^{|Q|} \, |\sum|$ directed edges. Now, the proof follows by taking B as the adjacency matrix of the deterministic digraph $DG = (2^Q, \sum, \delta)$ and Theorem 3.6 in [6]. □

Theorem 3.15. *Let* $(Q_i, \sum_i, \delta'_i), i = 1, 2, \ldots, n$ *be any* n *non-deterministic digraphs. Then, the deterministic matrix* A *of the product digraph*

$$\left(\prod_1^n 2^{Q_i}, \prod_1^n \sum_i, \delta\right)$$

of the deterministic digraphs $(2^{Q_i}, \sum_i, \delta_i)$ *is a square matrix of order* $\prod\limits_1^n 2^{|Q_i|}$ *with common row sum* $\prod\limits_1^n |\sum_i|$.

Proof. The proof follows by Theorems 3.7, 3.10 and Theorem 3.8 of [6]. □

Theorem 3.16. *Let* $A = (a_{ij})_{m\times m}$ *be a non-deterministic matrix of order* m *and* $n = \max\limits_{1\le i\le m}\left\{\sum\limits_1^m a_{ij}\right\}$. *Then the number of distinct deterministic matrices of order* 2^m *with common row sum* n *generated by* A *is*

$$\prod_{i=1}^m \left[\prod_{j=1}^m \delta_{ij} \begin{pmatrix} n - \sum\limits_{k=0}^{j-1} a_{ik} \\ a_{ij} \end{pmatrix}\right], \tag{4}$$

where $\begin{pmatrix} n \\ r \end{pmatrix} = \dfrac{n!}{(n-r)!r!}$ *and* $\delta_{ij} = \begin{cases} 1, & \text{if, } a_{ij} \neq 0\,, \\ 0, & \text{otherwise}\,. \end{cases}$

Proof. Let $(Q, \sum, \delta')$ and S_{ij} be defined as in Theorem 3.12. From (3), distinct selections of $S_{ij} \subset \sum$ defined in (2) generate distinct transition functions δ'. Hence, the number of distinct non-deterministic digraphs generated by A is the number of distinct selections of S_{ij} from $\sum$. Since $|\sum| = n$, for each pair, $(i, j) \in Q \times Q$, we can select S_{ij} in $\begin{pmatrix} n - \sum\limits_{k=0}^{j-1} a_{ik} \\ a_{ij} \end{pmatrix}$ ways if $a_{ij} \neq 0$. Now the proof follows from Theorem 3.7, Definition 2.5 and by taking $S_{i0} = \phi$. □

4. Complexity Metric on the Class of Non-Deterministic Matrices

In [3], Hirohisa Aman, H. Yamada, M.T. Noda and Y. Yanarau studied a graph – based class structural complexity metric and its evaluation. In [6], the authors established complexity metric and structural measure on the class of deterministic matrices. Hence, in this section we develop complexity metric and structural measure on the class of non-deterministic matrices.

Definition 4.1. Let $\mathbb{N}$ be the class of all non-deterministic matrices, $\mathbb{D}$ be the class of all deterministic matrices generated by the non-deterministic matrices of $\mathbb{N}$ and let $\mathbb{R}$ be the set of all real numbers. A complexity metric on $\mathbb{D}$ is a function $\mu : \mathbb{D} \to \mathbb{R}$ satisfying the following five properties.

(i) Non-negativity: $\mu(B) \geq 0$ for all $B \in \mathbb{D}$.

(ii) Null value: $\mu(B) = 0$ if B is the zero matrix in $\mathbb{D}$.

(iii) Similarity: $\mu(B) = \mu(C)$ if B and C are of the same order and $RS(B) = RS(C)$.

(iv) Super additivity: If B_1, B_2 and B are matrices of same order in $\mathbb{D}$ such that $RS(B_1) + RS(B_2) \leq RS(B)$, then $\mu(B_1) + \mu(B_2) \leq \mu(B)$.

(v) Additivity: If B_1, B_2 and B are in $\mathbb{D}$ such that $B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$, then $\mu(B_1) + \mu(B_2) = \mu(B)$.

By Theorem 3.14, for each $A \in \mathbb{N}$, there exists a deterministic matrix B generated by A, say $d(A) \in \mathbb{D}$. The function $\mu' : \mathbb{N} \to \mathbb{R}$ defined by $\mu'(A) = \mu(d(A))$ (by (iii) μ' is well defined) is called a complexity metric on the class of non-deterministic matrices.

Example 4.2. To each $r \in \mathbb{R}^+$, $\mu' : \mathbb{N} \to \mathbb{R}$ defined by $\mu'(A) = rRS(d(A))$ is a complexity metric on $\mathbb{N}$.

Definition 4.3. (see [6]) Let $A_{m\times m}$ be any non-deterministic matrix of order m and $NG = (Q, \sum, \delta')$ be a non-deterministic digraph generated by A, constructed by Theorem 3.12. Let $DG = (2^Q, \sum, \delta)$ be the deterministic digraph for the NG constructed by Theorem 3.7. Let $B_{2^m\times 2^m}$ be the deterministic matrix obtained by DG and $\sum\limits_{k=1}^{\infty} w_k > 0$. Then the matrix

$$B_w^* = \sum_{k=1}^{\infty} w_k \ B^k \tag{5}$$

is called the weighted closure of B. Let $\mathbb{P} = (B_w^*)'$, which is the transpose of B_w^*. Then $\mathbb{P}$ is called the dependence matrix of B. If $\mathbb{P} = (p_{ij})$ is a dependence matrix of order 2^m, then the vector

$$p_k = \left(\sum_{i=1}^{2^m} p_{ik}, \sum_{j=1}^{2^m} p_{kj} \right) \tag{6}$$

is called the dependency vector of the k-th vertex of the deterministic digraph $(2^Q, \sum, \delta)$. If $p = (p_1, p_2)$ is any dependency vector, then the norm of p is defined as

$$\|p\| = p_1 + p_2\,. \tag{7}$$

Let $c : \{1, 2, \dots, 2^m\} \to [0, \infty)$ be a non-negative real valued function. Define a function $SM : \mathbb{N} \to \mathbb{R}$ defined by

$$SM(A) = \sum_{k=1}^{2^m} \|p_k\|\, c(k) = SM(d(A))\,, \tag{8}$$

where $\|p_k\|$ is obtained by (6) and (7) with $B = d(A)$. By Theorems 2.7 and 3.14, (8) is well defined and is called the structural measure of the non-deterministic matrix A.

Lemma 4.4. *If A_1 and A_2 are two non-deterministic matrices of the same order, then $SM(A_1 + A_2) \geq SM(A_1) + SM(A_2)$. Also if maximum row sum n_1 of A_1 = maximum row sum n_2 of A_2, then $SM(A_1) = SM(A_2)$.*

Proof. The proof follows from Theorem 3.14 and the equations (5)-(8). □

Theorem 4.5. *Let $\mathbb{N}$ be the class of all non-deterministic matrices and $\mathbb{R}$ be the set of all real numbers. Then the function $\mu' : \mathbb{N} \to \mathbb{R}$ defined by $\mu'(A) = SM(A)$ for all $A \in \mathbb{N}$ is a complexity metric.*

Proof. The proof follows by Lemma 4.4, replacing A by $B = d(A)$, Q by $2^Q = \{1, 2, \dots, 2^m\}$, A_1, A_2 by $d(A_1), d(A_2)$ in Theorem 4.6 of [6]. □

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