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ON THE DOMINATION NUMBER OF KNÖDEL GRAPH $W_{3,n}$

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Abstract: Let $G=(V(G),E(G))$ be a graph. A set $S \subseteq V(G)$ is a dominating set if every vertex of $V(G)-S$ is adjacent to some vertices in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. In this paper, we study the domination number of Knödel graph $W_{3,n}$ and prove that for $n \geq 8$

$$\gamma(W(3,n)) = 2\lfloor \tfrac{n}{8} \rfloor + \begin{cases} 0, & n=0 \bmod 8, \\ 1, & n=2 \bmod 8, \\ 2, & n=4,6 \bmod 8. \end{cases}$$

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. A graph $G=(V(G),E(G))$ is a set $V(G)$ of vertices and a subset $E(G)$ of the unordered pairs of vertices, called edges. We use [8] for the terminology and notation not defined here.

The open neighborhood and the closed neighborhood of a vertex $v \in V$ are denoted by $N(v)=\{u \in V(G) \ : \ vu \in E(G)\}$ and $N[v]=N(v)\cup\{v\}$, respectively. For a vertex set $S \subseteq V(G)$, $N(S)=\bigcup_{v\in S} N(v)$ and $N[S]=\bigcup_{v\in S} N[v]$. The maximum degree of vertices in $V(G)$ is denoted by $\Delta(G)$.

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A set $S \subseteq V(G)$ is a dominating set if for each $v \in V(G)$ either $v \in S$ or v is adjacent to some $w \in S$. That is, S is a dominating set if and only if $N[S] = V(G)$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

The study of domination in graphs was initiated by Ore [12]. Topic on domination number and related parameters have long attracted graph theorists for their strongly practical background and theoretical interest. It has been proved [6] that the decision problem corresponding to the domination number for arbitrary graphs is NP-complete. So much work was done to establish bounds on $\gamma(G)$. There are the well known bounds on $\gamma(G)$ in terms of the number of vertices n and maximum degree $\Delta(G)$.

Theorem 1.1. (see [1], [13]) *For any graph* G, $\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma(G) \leq n - \Delta(G)$.

In 1995, Molloy and Reed [11] studied the domination number of a random cubic graph and proved $.2636n \leq \gamma(G) \leq .3126n$.

The domination numbers of very few families of graphs are known exactly. By [8], we have, $\gamma(K_n) = 1$, $\gamma(K_{1,n-1}) = 1 (n \geq 2)$, $\gamma(K_{m,n}) = 2 (m \geq 2, n \geq 2)$, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$.

The Cartesian product of two graphs G and H is the graph denoted $G \Box H$, with $V(G \Box H) = V(G) \times V(H)$ and $((u, u'), (v, v')) \in E(G \Box H)$ if and only if $u' = v'$ and $(u, v) \in E(G)$ or $u = v$ and $(u', v') \in E(H)$. The grid graph $G_{k,n} = P_k \Box P_n$.

In 1983, M.S. Jacobson and L.F. Kinch [9] determined the domination number $\gamma(G_{k,n})$ for $k \leq 4$. In 1993, T.Y. Chang and W.E. Clark [2] determined $\gamma(G_{k,n})$ for $5 \leq k \leq 6$. In 1993, D.C. Fisher determined $\gamma(G_{k,n})$ for $7 \leq k \leq 16$ and gave out the following conjecture [4]:

Conjecture 1.2. $\gamma(G_{m,n}) = \lfloor (m + 2)(n + 2)/5 \rfloor - 4$.

The cross product of two graphs G and H is the graph denoted $G \times H$, with $V(G \times H) = V(G) \times V(H)$ and $((u, u'), (v, v')) \in E(G \times H)$ if and only if $(u, v) \in E(G)$ and $(u', v') \in E(H)$.

In 1995, S. Gravier and A. Khelladi [7] determined the domination number $\gamma(P_n \times \overline{P_k})$ for every $n \geq 2$ and $k \geq 4$. In 1999, R. Chérifi, S. Gravier, and X. Lagraula et al [3] determined the domination number $\gamma(P_n \times P_k)$ for $k \leq 8$, $\gamma(P_n \times P_9)$ for $n \geq 8$ and $\gamma(P_n \times P_k)$ for $10 \leq k \leq 33$ and $1 \leq n \leq 40$.

In 1995, S. Klavžar and N. Seifter [10] determined the domination number $\gamma(C_n \Box C_k)$ for $k \leq 5$.

In 2006, Yang Yuansheng, Fu Xueliang, Jiang Baoqi [5] studied the generalized Petersen graph $\mathrm{P}(n,3)$ and determined the domination number $\gamma(\mathrm{P}(n,3)) = n - 2\lfloor \frac{n}{4} \rfloor (n \neq 11)$.

The Knödel graph $W_{\Delta,n}$ has even $n \geq 2$ vertices and degree Δ, $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$. The vertices of $W_{\Delta,n}$ are the pairs (i,j) with $i = 1,2$ and $0 \leq j \leq n/2 - 1$. For every j, $0 \leq j \leq n/2 - 1$, there is an edge between vertex $(1,j)$ and every vertex $(2,(j + 2^k - 1) \bmod (n/2))$, for $k = 0, ..., \Delta - 1$.

For $W_{\Delta,n}$, let v_j represent vertex $(1,j)$ and u_j represent vertex $(2,j)$. In this paper, the vertex labels are read modulo $n/2$ unless specified otherwise. By the definition of the Knödel graph, for $\Delta = 3$ and even $n \geq 8$, we have

$$\begin{aligned} V(W_{3,n}) &= \{v_0, v_1, \cdots, v_{n/2-1}, u_0, u_1, \cdots, u_{n/2-1}\}, \\ E(W_{3,n}) &= \bigcup_{t=0}^{n/2-1} \{v_t u_t, v_t u_{t+1}, v_t u_{t+3}\}. \end{aligned}$$

In this paper, we consider the Knödel graph $W_{3,n} (n \geq 8)$.

2. mathversionboldThe Domination Number of $W_{3,n}$

Let $m = \lfloor \frac{n}{8} \rfloor$ and $t = n \bmod 8$, then $n = 8m + t$. Since n is even, we have $t = 0, 2, 4, 6$.

Lemma 2.1. *For* $n \geq 8$,

$$\gamma(W(3,n)) \leq 2m + \begin{cases} 0, & t = 0, \\ 1, & t = 2, \\ 2, & t = 4, 6. \end{cases}$$

Proof. Let

$$S' = \begin{cases} \{v_{4i}, u_{4i+2} \ : \ 0 \leq i \leq m-1\}, & t = 0, \\ \{v_{4i}, u_{4i+2} \ : \ 0 \leq i \leq m-1\} \cup \{u_{4m}\}, & t = 2, \\ \{v_{4i}, u_{4i+2} \ : \ 0 \leq i \leq m-1\} \cup \{v_{4m}, u_{4m}\}, & t = 4, \\ \{v_{4i}, u_{4i+2} \ : \ 0 \leq i \leq m\}, & t = 6. \end{cases}$$

Then $N[S'] = V(W(3,n))$, S' is a dominating set of $W(3,n)$ with

$$|S'| = 2\lfloor \tfrac{n}{8} \rfloor + \begin{cases} 0, & t = 0, \\ 1, & t = 2, \\ 2, & t = 4, 6. \end{cases}$$

So,

$$\gamma(W(3,n)) \leq 2m + \begin{cases} 0, & t = 0, \\ 1, & t = 2, \\ 2, & t = 4, 6. \end{cases}$$

□

In Figure 2.1, we show the dominating sets of $W(3,n)$ for $8 \le n \le 14$, where the vertices of S' are in black.

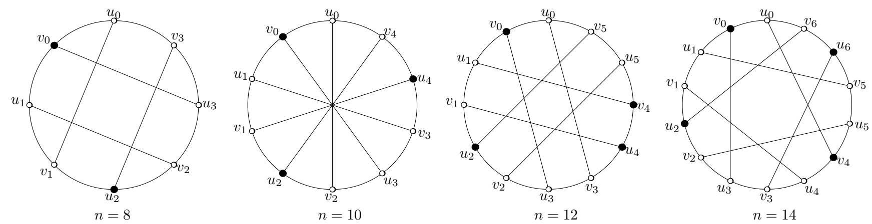

Figure 2.1.

Let S be an arbitrary dominating set of G. Each vertex $v \in V(G)$ is being dominated $|N[v] \cap S|$ times. We define the function rd counting the times v is re-dominated as follows:

$$rd(v) \quad = |N[v] \cap S| - 1.$$

For a vertex set $V' \subseteq V(G)$, let $rd(V') = \sum_{v \in V'} rd(v)$. Clearly, for an arbitrary k-regular graph G with order n, we have $rd(V(G)) = (k+1) \times |S| - n$.

Lemma 2.2. *For* $n \ge 8$,

$$\gamma(W(3,n)) \ge 2m + \begin{cases} 0, & t=0, \\ 1, & t=2, \\ 2, & t=4,6. \end{cases}$$

Proof. Since the Knödel Graph $W_{3,n}$ is 3-regular graph, by Theorem 1.1, we have $\gamma(W(3,n)) \ge \lceil \frac{n}{3+1} \rceil = \lceil \frac{8m+t}{4} \rceil = 2m + \lceil \frac{t}{4} \rceil$. Hence, we have

$$\gamma(W(3,n)) \ge 2m + \begin{cases} 0, & t=0, \\ 1, & t=2,4, \\ 2, & t=6. \end{cases}$$

Assume there exists a dominating set of $W(3,n)$, say S^*, with $|S^*| = 2m+1$ for $t=4$, we have $rd(W(3,n)) = 4 \times |S^*| - (8m+4) = 4 \times (2m+1) - (8m+4) = 0$, i.e. $rd(w) = 0$ for any vertex w in $V(W(3,n))$. Without loss of generality, let $v_0 \in S^*$. Consider the vertex v_1, since $N[S^*] = V(W(3,n))$, we have $N[v_1] \cap S^* \ne \emptyset$. i.e. $\{u_1, v_1, u_2, u_4\} \cap S^* \ne \emptyset$. Since $rd(u_1) = 0$, $v_0 \in S^*$, we have $u_1, v_1 \notin S^*$, hence there is at least one vertex of $\{u_2, u_4\}$ which belongs to S^*.

Suppose $u_4 \in S^*$(see Figure 2.2 (1)). Since $rd(u_3) = 0$, $v_0 \in S^*$, we have $v_2, u_3 \notin S^*$. Since $rd(v_1) = 0$, $u_4 \in S^*$, we have $u_2 \notin S^*$. Since $rd(v_4) = 0$, $u_4 \in S^*$, we have $u_5 \notin S^*$. So $\{u_2, v_2, u_3, u_5\} \cap S^* = \emptyset$, i.e. $N[v_2] \cap S^* = \emptyset$, a contradiction with $N[S^*] = V(W(3,n))$, so $u_4 \notin S^*$. It forces that $u_2 \in S^*$.

Consider the vertex v_3, since $N[v_3] \cap S^* \neq \emptyset$, we have $\{u_3, v_3, u_4, u_6\} \cap S^* \neq \emptyset$. Since $rd(u_3) = 0$, $v_0 \in S^*$, we have $u_3, v_3 \notin S^*$. Since $rd(v_1) = 0$, $u_2 \in S^*$, we have $u_4 \notin S^*$. It forces $u_6 \in S^*$. Since $N[u_5] \cap S^* \neq \emptyset$, we have $\{v_2, v_4, u_5, v_5\} \cap S^* \neq \emptyset$. Since $rd(v_5) = 0$, $u_6 \in S^*$, we have $u_5, v_5 \notin S^*$. Since $rd(v_2) = 0$, $u_2 \in S^*$, we have $v_2 \notin S^*$. It forces $v_4 \in S^*$. Continuing in this way, we have $\{v_{4i}, u_{4i+2}\} \subset S^*(0 \leq i \leq m-1)$, i.e. $v_{4m-4}, u_{4m-2} \in S^*$. Since $rd(u_0) = 0$, $v_0 \in S^*$, we have $v_{4m-1}, u_0 \notin S^*$. Since $rd(v_{4m-2}) = 0$, $rd(v_{4m-3}) = 0$, and $u_{4m-2} \in S^*$, we have $u_{4m-1}, u_{4m} \notin S^*$, i.e. $N[v_{4m-1}] \cap S^* = \emptyset$, a contradiction with $N[S^*] = V(W(3,n))$(see Figure 2.2 (2)). Hence $|S^*| \neq 2m+1$, $|S^*| \geq 2m+2$, i.e. $\gamma(W(3,n)) \geq 2m+2$ for $t = 4$. □

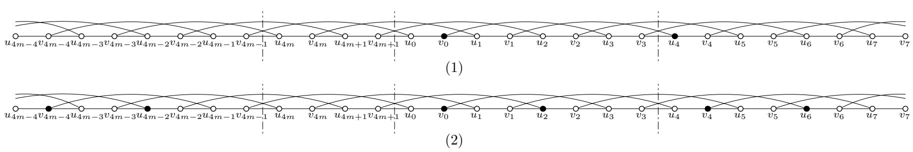

Figure 2.2.

From Lemmas 2.1-2.2, we have

Theorem 2.3. *For $n \geq 8$,*

$$\gamma(W(3,n)) = 2\lfloor \tfrac{n}{8} \rfloor + \begin{cases} 0, & n = 0 \bmod 8, \\ 1, & n = 2 \bmod 8, \\ 2, & n = 4, 6 \bmod 8. \end{cases}$$

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References

[1] C. Berge, *Theory of Graphs and its Applications*, Methuen, London (1962).

[2] T.Y. Chang, W.E. Clark, The domination number of the 5×n and the 6×n grid graphs, *J. Graph Theory*, **17**, No. 1 (1993), 81-107.

[3] R. Chérifi, S. Gravier, X. Lagraula et al, Domination number of the cross product of paths, *Discrete Applied Mathematics*, **94** (1999), 101-134.

[4] D.C. Fisher, The domination number of complete grid graphs, *Preprint* (1993).

[5] Xueliang Fu, Yuansheng Yang, Baoqi Jiang, On the domination number of generalized petersen graph $\mathrm{P}(n, 3)$, *Ars Combin.*, To Appear.

[6] M.R. Garey, D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman, San Francisco (1979).

[7] S. Gravier, A. Khelladi, On the domination number of cross products of graphs, *Discrete Math.*, **145** (1995), 273-277.

[8] T.W. Haynes, S.T. Hedetneimi, P.J. Slater, *Domination in Graphs – Advanced Topics*, Marcel Dekker, Inc., New York (1998).

[9] M.S. Jacobson, L.F. Kinch, On the domination number of products of graphs, *I. Ars Combin.*, **18** (1983), 33-44.

[10] S. Klavžar, N. Seifter, Dominating Cartesian products of cycles, *Discrete Applied Mathematics*, **59** (1995), 126-136.

[11] M. Molloy, B. Reed, The dominating number of a random cubic graph, *Random Structures and Algorithms*, **7** (1995), 209-221.

[12] O. Ore, Theory of graphs, *Amer. Math. Soc. Colloq. Publ.*, **38**, Providence (1962).

[13] H.B. Walikar, B.D. Acharya, E. Sampathkumar, Recent developments in the theory of domination in graphs, In: *MRI Lecture Notes in Math.*, Mahta Resarch Instit., Allahabad, Volume 1 (1979).