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HARMONIC GRADIENTS, HÖLDER NORMS FOR ELLIPTIC FUNCTIONS, AND SOLUTIONS TO POISSON'S EQUATION ON A NONSMOOTH DOMAIN

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Abstract: Solutions to $Lu = \operatorname{div} \overrightarrow{f}$ in a bounded, nonsmooth domain Ω, $u = g$ on $\partial\Omega$, are investigated using a local Hölder norm of u and different measures on Ω. Results for 2-nd order strictly elliptic operators are presented, and problems that arise in proving similar theorems for their parabolic counterparts are discussed.

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1. Introduction

My work will center on some problems arising from the question of how the rate of change of a generalized temperature function or of a solution to an elliptic equation, can be shown to depend on the temperature at which the boundary of a domain is kept (specified boundary data) and/or by external heat sources within the domain. For the steady state heat equation with no external sources this means examining $\nabla u(x)$, where u is a harmonic function, i.e. $\Delta u(x) = 0$

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in Ω and $u\,|_{\partial\Omega}\,(x') = g(x')$.

With external sources the model can be given as $\Delta u(x) = \operatorname{div}\overrightarrow{f}(x)$ in Ω and $u\,|_{\partial\Omega} = g(x')$.

There are two generalizations that I will be dealing with; the first is to have a time-dependent temperature, i.e. a solution to $(\partial/\partial t - \Delta)u(x,t) = \operatorname{div}\overrightarrow{f}(x,t)$, with boundary values for u being specified on the parabolic boundary of the domain. The second generalization is to replace Δ by L, a divergence form operator which is symmetric and strictly elliptic (or strictly parabolic for $\partial/\partial t - L$). This means that

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{i,j}(x)\frac{\partial}{\partial x_j}) \quad \text{or} \quad L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{i,j}(x,t)\frac{\partial}{\partial x_j}),$$

with $a_{i,j}(x) = a_{j,i}(x)$, $a_{i,j}(x)$ bounded and measurable on Ω, and there exists a constant $\lambda > 0$ such that $\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{n} \xi_i a_{i,j}(x)\xi_j \leq \lambda|\xi|^2$ (or $\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{n} \xi_i a_{i,j}(x,t)\xi_j \leq \lambda|\xi|^2$ for the time-dependent equation).

In sum I am concerned with solutions to

$$Lu(x) = \operatorname{div}\overrightarrow{f}(x) \text{ in } \Omega \text{ and } u\,|_{\partial\Omega}\,(x') = g(x')$$

and to

$$\begin{aligned} (\partial/\partial t - L)u(x,t) &= \operatorname{div}\overrightarrow{f}(x,t) \text{ in } \Omega_T \\ \text{with } u\,\big|_{\partial_p\Omega_T}\,(X') &= g(X'),\ X' \in \partial_p\Omega_T \end{aligned}$$

if $X' = (x',t)$, a point on the lateral boundary of Ω_T, or if $X' = (x,0)$, a point on the bottom of the domain Ω_T. Usually the domain $\Omega \subset \mathbb{R}^n$ or $\Omega_T \subset \mathbb{R}^{n+1}$ will be bounded, although occasionally I will refer to results for Ω being the upper half space $\{(x',x_n) : x_n > 0\}$, or the right half space for Ω_T, $\{(x,t) : x_n > 0\}$.

One way to investigate the full problem is to break it into two separate problems; then superposition gives a solution to the full problem. The first is the Dirichlet problem mentioned above, namely

$$Lu(x) = 0 \text{ in } \Omega \text{ and } u\,|_{\partial\Omega}\,(x') = g(x').$$

The second problem is Poisson's equation

$$Lu(x) = \operatorname{div}\overrightarrow{f}(x) \text{ in } \Omega \text{ and } u\,|_{\partial\Omega}\,(x') = 0.$$

Specifically, for the Dirichlet problem I am interested in finding conditions on two measures μ a Borel measure on Ω (or on Ω_T), and a second measure $\nu d\omega$, a measure on $\partial\Omega$ composed of a non-negative weight $\nu(x')$ multiplied by the "harmonic" measure of the domain $d\omega(x')$, so that

$$\left(\int_\Omega |\nabla u(x)|^q \, d\mu(x)\right)^{1/q} \leq C\left(\int_{\partial\Omega} \left|g(x')\right|^p \nu(x') d\omega(x')\right)^{1/p}. \tag{1}$$

For Poisson's equation the question is: what measures μ and η, both Borel measures on Ω, can give the norm inequality

$$\left(\int_\Omega |\nabla u(x,t)|^q \, d\mu(x,t)\right)^{1/q} \leq C\left(\int_\Omega \left(\left|\operatorname{div}\overrightarrow{f}(x)\right|^p + \left|\overrightarrow{f}(x)\right|^p\right) d\eta(x)\right)^{1/p}?$$

Of course we want these norm bounds to be valid for as large a range of q and p as possible.

Last summer at this conference I announced the following theorem for the elliptic version of Poisson's equation.

Theorem 1. (see [12]) *For $3 \leq n < s < p \leq q < \infty$, if for any $Q_j \in \mathcal{W}$,*

$$\mu(Q_j)^{1/q} M(Q_j) l(Q_j)^{-n-\alpha} \leq C_0$$

then for any u, a solution to $Lu = \operatorname{div}\overrightarrow{f}$ in Ω, $u\,|_{\partial\Omega} = 0$, there is a constant C independent of u, $\overrightarrow{f}$, μ and η so that

$$\left(\int_\Omega \left(\|u\|_{H^\alpha}(x)\right)^q d\mu(x)\right)^{1/q} \leq C\left(\int_\Omega \left(\left|\overrightarrow{f}(x)\right|^p + \left|div\,\overrightarrow{f}(x)\right|^p\right) d\eta(x)\right)^{1/p}. \tag{2}$$

Remark. By allowing C to depend on $\mu(\Omega)$, on $\eta(\Omega)$ and on q_0 and p_0, the range of p and of q can be extended to $0 < q \leq q_0$ and $n < s < p_0 \leq p < \infty$ for any fixed pair of indices p_0 and q_0 having $s < p_0 \leq q_0$. This follows using Hölder's inequality on both integrals in (2).

The relevant definitions are:

i. $\mathcal{W}$ is a collection of dyadic cubes that are Whitney-type with respect to Ω, i.e. if $Q \in \mathcal{W}$, then $Q \subset \Omega$, and $diam(Q) \sim l(Q) \sim distance(Q, \partial\Omega)$. $l(Q_j) =$ the side length of Q_j.

ii. $M(Q_j)$: for any dyadic cube Q_j that lies inside Ω, and for $d\sigma(y) = \left(\frac{d\eta}{dy}(y)\right)^{1-p'} dy$,

$$M(Q_j) = \max\{\left(\frac{1}{|Q_j|}\int_{4Q_j}\left(\frac{d\eta}{dx}(x)\right)^{s/(s-p)} dx\right)^{1/s-1/p} \cdot l(Q_j)^{n/p'+1},$$

$$\left(\int_{Q_0}\left(1+\frac{|y-x_{Q_j}|}{l(Q_j)}\right)^{-(n-\epsilon)p'/2} d\sigma(y)\right)^{1/p'} \}.$$

iii. $\|u\|_{H^\alpha}(x) = \sup_{|x-y|<.01\delta(x)} \left(\frac{|u(y)-u(x)|}{|x-y|^\alpha}\right)$, with $\delta(x) = \text{distance}\,(x, \partial\Omega)$.

iv. As above, L is a strictly elliptic operator with coefficients symmetric, bounded and measurable. Ω is a bounded domain in $\mathbb{R}^n$ that satisfies and exterior cone condition. The prototypical such domain is a Lipschitz domain.

2.

I will discuss some refinements in the proof of this theorem, and progress I have made on proving a version for this result for solutions to a parabolic equation. Unfortunately I do not know whether the natural extension of Theorem 1 to the time dependent situation is true.

Of course it would have been desirable to retain $|\nabla u(x)|$ instead of the Hölder norm $\|u\|_{H^\alpha}(x)$ as the integrand on the left hand side of (2); for a general elliptic function, however, one must add too many new restrictions on the measures and on the range of indices to prove the norm inequality. The result is much better for a local Hölder norm. One would also have liked to have Lebesgue surface measure in the integral on the right in (1) instead of elliptic measure; again, this is not feasible unless $\partial\Omega$ is so smooth that it satisfies an A^∞ condition with respect to the harmonic/elliptic measure (this depends on the smoothness of L as well). Since I want to investigate non-smooth domains, $d\omega$ is the natural measure to use.

When $L = \Delta$, the author and J. M. Wilson, building on the work of Wheeden and Wilson [17], have proved sufficient conditions so that (1) is valid for Ω being a bounded Lipschitz domain. They used the dual operator method with a square function inequality; this was the method employed by Wheeden and Wilson when they found necessary and sufficient conditions on measure so that (1) is valid for harmonic u in the upper half plane.

It turns out that one can use the same general approach to proving Theorem 1 that Wheeden and Wilson [17], and later Sweezy and Wilson [16] used to obtain (1) ; i.e. a dual operator inequality coupled with a square function theorem. And, for both the Dirichlet problem solutions and solutions to Poisson's equation, most of the work comes in proving the right square function inequality. The groundwork for these results was done by Wilson in [18].

In fact, I might mention here that there is a completely separate question of weighted square function inequalities. J. M. Wilson [19] has recently published a book that gives thorough treatment of this subject However, for the purpose of proving Theorem 1, we do not need weights in the square function inequalities.

The square function theorem which I state below, is independent of the theorems on the pde solutions; however, to have a result that is useful in proving inequalities such as (1) and (2), one needs to make sure that theorem will apply to the particular kind of function that arises from using the dual operator method. These functions have the form $f(x) = \sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$, with the J being dyadic cubes, $\mathcal{F}$ being a finite family of such cubes. The $\varphi_{(J)}(x)$ are functions that satisfy certain decay, smoothness and cancellation conditions. In fact, it is in verifying the degree of smoothness for the $\varphi_{(J)}(x)$ that the refinement of the proof for (2) and the barrier to proving an equivalent result for parabolic solutions arises, so, after stating the square function theorem for $\Omega \subset \mathbb{R}^n$, I will discuss these topics.

When $f(x) = \sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$, let

$$g^*(x) = \left(\sum_{J\in\mathcal{F}} \frac{\lambda_J^2}{|J|} \left(1 + \frac{|x - x_J|}{l(J)}\right)^{-n+\epsilon} \right)^{1/2}.$$

Theorem 2. (see [12]) *Suppose that* $f(x) = \sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$ *is a function defined on* Ω, *where* $\mathcal{F}$ *is a finite set of dyadic cubes from* $\mathcal{W}$, *and the* $\{\varphi_{(J)}\}_{J\in\mathcal{F}}$ *are a family of functions that satisfy conditions* **a**), **a′**), **b**), *and* **c**), *and such that* $\varphi_{(J)}(x) = 0$ *if* $x \in Q_0 \setminus \Omega$. *Then, if* $d\sigma \in A^\infty(Q_0, dx)$, *there is a constant* $C = C(n, \alpha, p, \Omega, \kappa, C_0)$ *such that, for any* $0 < p < \infty$,

$$\|f\|_{L^p(Q_0,d\sigma)} \le C \|g^*\|_{L^p(Q_0,d\sigma)}. \tag{3}$$

The conditions mentioned in the theorem are:

a) $\left|\varphi_{(J)}(x)\right| \le C l(J)^{2-n/2} \left(1 + \frac{|x - x_J|}{l(J)}\right)^{2-n}$ for all $x \in \Omega$.

a′) $\left|\varphi_{(J)}(x)\right| \le C\delta(x)^\alpha l(J)^{2-n/2-\alpha} \left(1 + \frac{|x - x_J|}{l(J)}\right)^{2-n-\alpha}$ for all $x \in \Omega$.

b) $\left|\varphi_{(J)}(x) - \varphi_J(y)\right| \le C |x - y|^\alpha l(J)^{2-n/2-\alpha}$.

$$\left(1+\frac{|x-x_J|}{l(J)}+\frac{|y-x_J|}{l(J)}\right)^{2-n-\alpha} \text{ for all } x,\, y \text{ in } \eta Q_j \text{ and } J\in\mathcal{S}(Q_j), Q_j\in\mathcal{D}.$$

$$\textbf{c)}\quad \int\left|\sum_{J\in\mathcal{F}}\lambda_J\varphi_{(J)}(x)\right|^2 dx \le C\sum_{J\in\mathcal{F}}\lambda_J^2.$$

For the parabolic version we need:

$$\textbf{i)}\quad \left|\varphi_{(J)}(x,t)\right| \le Cl(J)^{1-n/2}\left(1+\frac{d_p(x,t;x_J,t_J)}{l(J)}\right)^{-n} \text{ for all } x,t\in\Omega_T.$$

$$\textbf{ii)}\quad \left|\varphi_{(J)}(x,t)\right| \le C\delta(x,t)^{\alpha}l(J)^{1-n/2-\alpha}\left(1+\frac{d_p(x,t;x_J,t_J)}{l(J)}\right)^{-n-\alpha} \text{ for all } x,t \text{ in a neighborhood of } S_T.$$

$$\textbf{iii)}\quad \left|\varphi_{(J)}(x,t)-\varphi_J(y,s)\right| \le Cd_p(x,t;y,s)^{\alpha}l(J)^{1-n/2-\alpha}\cdot$$

$$\left(1+\frac{d_p(x,t;x_J,t_J)}{l(J)}+\frac{d_p(y,s;x_J,t_J)}{l(J)}\right)^{-n-\alpha} \text{ for certain } x,t \text{ and } y,s.$$

$$\textbf{iv)}\quad \int\left|\sum_{J\in\mathcal{F}}\lambda_J\varphi_{(J)}(x,t)\right|^2 dxdt \le C\sum_{J\in\mathcal{F}}\lambda_J^2.$$

These conditions are used to prove a series of local inequalities for functions that "stand in" for $f(x)$ and $g^*(x)$; the inequalities then give a good-λ inequality that translates to be **3**) by a standard argument.

The φ_J are, in the case of $Lu=0$, $\varphi_{(Q)}(y')=l(Q)^{1-\alpha/2}\sqrt{\omega(Q)}K(X_{T(Q)},y')$ (if u is harmonic, then $\alpha=0$). $K(x,y')$ is a kernel function for L on Ω. The existence, uniqueness and geometric properties of $K(x,y')$ were proved by [2] and [8] quite some time ago (see [9]). These are sufficient to obtain properties **a**), **a′**), **b**), and **c**) for these $\varphi_{(Q)}$. In the case of $Lu=\operatorname{div}\overrightarrow{f}$, the $\varphi_{(J)}$ have a different form: $\varphi_{(J)}(y)=\frac{1}{\sqrt{|J|}}\int_{(3/2)J}(G(x,y)-\widetilde{G}(x,y))dx$, where $G(x,y)$ is the Green's function generated by L on the domain Ω, and $\widetilde{G}(x,y)$ is the Green's function of the cube $4J$, also generated by L. Estimates of Grüter and Widman [7] on the Green's function of a bounded domain that satisfies an exterior cone condition, along with the maximum principle, Harnack's inequality and the standard interior Hölder continuity for non-negative solutions to any strictly elliptic pde, are sufficient to prove **a**), **a′**), **b**), and **c**). There is a technical hitch that occurs because the cube $4J$ is not one of the dyadic cubes in the family $\mathcal{D}$, however, a case by case analysis shows that the necessary estimates hold.

For the parabolic equation $(\partial/\partial t - L)u(x,t) = \operatorname{div} \overrightarrow{f}(x,t)$ *in* Ω_T , $\varphi_{(J)}(y,s) = \frac{1}{\sqrt{|J|}} \int_{(3/2)J} (G(x,t;y,s) - \widetilde{G}(x,t;y,s))dxdt$. Here J is a parabolic dyadic cube: its time dimension is $l(J)^2$ when the space dimension is $l(J)$. This time the Green's function satisfies geometric decay so **i**) follows easily. **iv**) is also not hard to verify. The problem is with **ii**) and **iii**); as long as the two cubes in question, J and Q have a certain distance from each other as far as the time variable goes, one can obtain the required estimates from interior Hölder continuity and Hölder continuity at the boundary. If they are very close together - almost on top of each other, one can also use the same kind of argument that works in the elliptic case. However, for the centers of Q and J separated, but almost identical in the time variable, the available version of Hölder continuity [1], [9] is not strong enough to prove either **ii**) and **iii**). Every other part of the proof, based on duality and the square function theorem, goes through without any more than the usual technical difficulties that occur in dealing with the parabolic setting.

References

[1] D. Aronson, Non-negative solutions of linear parabolic equations, *Ann. Scuola Norm. Sup. Pisa*, **22** (1968), 607-694.

[2] L. Caffarelli, E. Fabes, Mortola, S. Salsa, Boundary behavior of non-negative solutions of elliptic operators in divergence form, *Indiana U. Math. J.*, **30** (1981), 621-640.

[3] R. Coifman, C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, *Stud. M.*, **51** (1974), 241-250.

[4] E. Fabes, N. Garofalo, S. Salsa, A backward Harnack inequality and Fatou theorems for non-negative solutions of parabolic equations, *I.J.M.*, **30** (1986), 536-565.

[5] E. Fabes, M. Safanov, Behavior near the boundary of positive solutions of second order parabolic equations, *J.F.A.A.*, **3** (1997), 871-82.

[6] M. Giaquinta, M. Struwe, On the partial regularity of weak solutions of non-linear parabolic systems, *Math. Zeitschrift*, **179** (1982), 437-451.

[7] M. Grüter, K-O. Widman, The Green function for uniformly elliptic equations, *Manuscripta Mathematica*, **37** (1982), 303-342.

[8] D. Jerison, C. Kenig, Boundary behavior of harmonic functions in nontangentially accessible domains, *Advances in Math.*, **146** (1982), 80-147.

[9] C. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, AMS, Providence (1994).

[10] K. Nystrom, The Dirichlet problem for second order parabolic operators, *Indiana U. Math. J.*, **46**, No. 1 (1997), 183-245.

[11] C. Sweezy, Weights and Hölder norms for solutions to a second order elliptic Dirichlet problem on nonsmooth domains, *AMS Con. Math.*, **428**, Harmonic Analysis, PDE, and Rel. Top. (2007), 127-138.

[12] C. Sweezy, Weighted norm bounds for a local Hölder norm of elliptic and of parabolic functions on a non-smooth domain in Euclidean space, *IJPAM*, **42**, No. 2 (2008), 183-190.

[13] C. Sweezy, A littlewood Paley type inequality with applications to the elliptic Dirichlet problem, *Ann. Polonici Math.*, **90**, No. 2 (2007), 105-130.

[14] C. Sweezy, Weighted norm inequalities for parabolic gradients on nonsmooth domains, *IJPAM*, **36**, No. 1 (2007), 41-62.

[15] C. Sweezy, Gradient norm inequalities for weak solutions to parabolic equations on bounded domains with and without weights, *WSEAS Transactions on Systems*, **4**, No. 12 (December 2005), 2196-2203.

[16] C. Sweezy, J. M. Wilson, Weighted inequalities for gradients on nonsmooth domains, Preprint.

[17] R.L. Wheeden, J.M. Wilson, Weighted norm estimates for gradients of half-space extensions, *Ind. U. Math. J.*, **44** (1995), 917-969.

[18] J.M. Wilson, Global orthogonality implies local almost orthogonality, *Revista Mathematica Iberoamericana*, **16** (2000), 29-48.

[19] J.M. Wilson, *Weighted Littlewood-Paley Theory and Exponential Sqaure Integrability*, Lecture Notes in Mathematics, Springer (2008).