International Journal of Pure and Applied Mathematics

Volume 51 No. 2 2009, 233-238

A NOTE ON THE TYPE OF A MEROMORPHIC FUNCTION OF ORDER INFINITY

Sanjib Kumar Datta [§]

Department of Mathematics
University of North Bengal
Raja Rammohunpur, District Darjeeling
PIN Code 734013, West Bengal, INDIA
e-mail: sanjib_kr_datta@yahoo.co.in

Abstract: In this paper we introduce the definition of the type of a meromorphic function of order infinity and discuss some growth properties of it.

AMS Subject Classification: 30D35, 30D30
Key Words: meromorphic function, entire function, infinite order, type, growth

1. Introduction, Definitions and Notations

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [4] and [3]. In the sequel we use the following notations:

$\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$, and

$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\exp^{[0]} x = x$.

The order ρ_f of a meromorphic function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

For $0 < \rho_f < \infty$, the type σ_f of f is defined in the following manner:

Received: December 5, 2008

[§]Correspondence address: 25, School Road, Kalianibash, Barrackpore, P.O.: Nonachandanpukur, Dist.: 24 Parganas (North), P.S.: Titagarh, PIN Code: 743102, West Bengal, INDIA

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}.$$

In case of an entire function f the above two definitions take the form:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}},\ 0 < \rho_f < \infty.$$

When a meromorphic or an entire function f is of infinite order, we cannot define the type of it. To overcome this situation we need the following definition.

Definition 1. The hyper order $\overline{\rho}_f$ of a meromorphic function f is defined by

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

When f is entire, one can easily verify that

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

With the help of Definition 1 we introduce the following definition of type of a meromorphic function of infinite order.

Definition 2. For a meromorphic function f of infinite order we define the type of f by

$$\overline{\sigma}_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{r^{\overline{\rho}_f}},\ \text{where } 0 < \overline{\rho}_f < \infty.$$

For example, taking $f = \exp^{[2]} z$ we see that $\rho_f = \infty$, $\overline{\rho}_f = 1$ and $\overline{\sigma}_f = 1$.

However the equivalence between the classical definition of the type of a meromorphic function of order infinity and its integral representation is proved in [1].

In the paper we establish some results on the growth properties relating to the type of a meromorphic function with infinite order.

2. A Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 1. (see [2]) *Let f be meromorphic and g be transcendental entire. If $\rho_{f\circ g} < \infty$ then $\rho_f = 0$.*

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f be a meromorphic function of infinite order and g be a transcendental entire function such that: $(i)\, 0 < \overline{\rho}_f < \infty$, $(ii)\, 0 < \overline{\rho}_{f\circ g} < \infty$, $(iii)\, 0 < \overline{\sigma}_{f\circ g} < \infty$ and $(iv)\, 0 < \overline{\sigma}_f < \infty$. Then:*

(i) $\liminf\limits_{r\to\infty} \frac{\log T(r,f\circ g)}{\log T(r,f)} \leq \frac{\overline{\sigma}_{f\circ g}}{\overline{\sigma}_f} \leq \limsup\limits_{r\to\infty} \frac{\log T(r,f\circ g)}{\log T(r,f)}$ *if* $\overline{\rho}_{f\circ g} = \overline{\rho}_f$.

(ii) $\limsup\limits_{r\to\infty} \frac{\log T(r,f\circ g)}{\log T(r,f)} = \infty$ *if* $\overline{\rho}_{f\circ g} > \overline{\rho}_f$.

(iii) $\liminf\limits_{r\to\infty} \frac{\log T(r,f\circ g)}{\log T(r,f)} = 0$ *if* $\overline{\rho}_{f\circ g} < \overline{\rho}_f$.

Proof. If possible let $\rho_{f\circ g} < \infty$. Then $\rho_f = 0$, $\{cf.\,[2]\}$ which contradicts the fact that the order of f is infinite.

(i) For arbitrary positive ε and for all sufficiently large values of r we obtain that

$$\log T\,(r, f\circ g) \leq (\overline{\sigma}_{f\circ g} + \varepsilon)\, r^{\overline{\rho}_{f\circ g}}. \tag{1}$$

Also for a sequence of values of r tending to infinity

$$\log T\,(r, f\circ g) \geq (\overline{\sigma}_{f\circ g} - \varepsilon)\, r^{\overline{\rho}_{f\circ g}}. \tag{2}$$

Similarly for all sufficiently large values of r we get

$$\log T\,(r, f) \leq (\overline{\sigma}_f + \varepsilon)\, r^{\overline{\rho}_f} \tag{3}$$

and for a sequence of values of r tending to infinity

$$\log T\,(r, f) \geq (\overline{\sigma}_f - \varepsilon)\, r^{\overline{\rho}_f}. \tag{4}$$

Now combining (1) and (4) it follows for a sequence of values of r tending to infinity that

$$\frac{\log T\,(r, f\circ g)}{\log T\,(r, f)} \leq \frac{(\overline{\sigma}_{f\circ g} + \varepsilon)\, r^{\overline{\rho}_{f\circ g}}}{(\overline{\sigma}_f - \varepsilon)\, r^{\overline{\rho}_f}}. \tag{5}$$

As $\varepsilon\,(> 0)$ is arbitrary, in view of the condition $\overline{\rho}_{f\circ g} = \overline{\rho}_f$ we obtain from (5) that

$$\liminf_{r\to\infty} \frac{\log T\,(r, f\circ g)}{\log T\,(r, f)} \leq \frac{\overline{\sigma}_{f\circ g}}{\overline{\sigma}_f}. \tag{6}$$

Again combining (2) and (3) we get for a sequence of values of r tending to infinity that

$$\frac{\log T\,(r, f\circ g)}{\log T\,(r, f)} \geq \frac{(\overline{\sigma}_{f\circ g} - \varepsilon)\, r^{\overline{\rho}_{f\circ g}}}{(\overline{\sigma}_f + \varepsilon)\, r^{\overline{\rho}_f}}. \tag{7}$$

Now in view of the condition $\overline{\rho}_{f\circ g} = \overline{\rho}_f$ and as $\varepsilon\,(>0)$ is arbitrary, it follows from (7) that

$$\limsup_{r\to\infty}\frac{\log T\,(r,f\circ g)}{\log T\,(r,f)} \geq \frac{\overline{\sigma}_{f\circ g}}{\overline{\sigma}_f}. \tag{8}$$

From (6) and (8) we obtain that

$$\liminf_{r\to\infty}\frac{\log T\,(r,f\circ g)}{\log T\,(r,f)} \leq \frac{\overline{\sigma}_{f\circ g}}{\overline{\sigma}_f} \leq \limsup_{r\to\infty}\frac{\log T\,(r,f\circ g)}{\log T\,(r,f)}.$$

This proves the first part of the theorem.

(ii) If we take the condition $\overline{\rho}_{f\circ g} > \overline{\rho}_f$ and as $\varepsilon\,(>0)$ is arbitrary, it follows from (7) that

$$\limsup_{r\to\infty}\frac{\log T\,(r,f\circ g)}{\log T\,(r,f)} = \infty.$$

Thus the second part of the theorem is established.

(iii) If we consider $\overline{\rho}_{f\circ g} < \overline{\rho}_f$ and as $\varepsilon\,(>0)$ is arbitrary, it follows from (5) that

$$\liminf_{r\to\infty}\frac{\log T\,(r,f\circ g)}{\log T\,(r,f)} = 0.$$

This establishes the third part of the theorem. □

Remark 1. The inequality in the first part of the theorem is best possible in the sense that '$\leq$' cannot be replaced by '$<$' only as we see in the following example.

Example 1. Let $f = \exp^{[2]} z$ and $g = z$. Then

$T\,(r,f) = T\,(r,f\circ g) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}$.

So

$$\liminf_{r\to\infty}\frac{\log T\,(r,f\circ g)}{\log T\,(r,f)} = 1 = \limsup_{r\to\infty}\frac{\log T\,(r,f\circ g)}{\log T\,(r,f)}.$$

Also $\overline{\rho}_f = \overline{\sigma}_f = \overline{\rho}_{f\circ g} = \overline{\sigma}_{f\circ g} = 1$.

In the line of Theorem 1 we may state the following theorem without proof.

Theorem 2. *Let f be a meromorphic function and g be a transcendental entire function satisfying the following conditions:*

$(i)\; 0 < \overline{\rho}_g < \infty, (ii)\; 0 < \overline{\rho}_{f\circ g} < \infty, (iii)\; 0 < \overline{\sigma}_{f\circ g} < \infty$ *and* $(iv)\; 0 < \overline{\sigma}_g < \infty$. *If* $\overline{\rho}_{f\circ g} = \overline{\rho}_g$ *then*

$$\liminf_{r\to\infty}\frac{\log T\,(r,f\circ g)}{\log T\,(r,g)} \leq \frac{\overline{\sigma}_{f\circ g}}{\overline{\sigma}_g} \leq \limsup_{r\to\infty}\frac{\log T\,(r,f\circ g)}{\log T\,(r,g)}.$$

Remark 2. Considering $f = z$ and $g = \exp^{[2]} z$ one can easily verify that the inequality in Theorem 2 is best possible.

Theorem 3. *Let f be a meromorphic function and g be a transcendental entire function such that $(i)\, 0 < \rho_f < \infty$, $(ii)\, 0 < \sigma_f < \infty$, $(iii)\, \rho_{f\circ g} = \infty$, $(iv)\, 0 < \overline{\rho}_{f\circ g} < \infty$, $(v)\, 0 < \overline{\sigma}_{f\circ g} < \infty$ and $(vi)\, \overline{\rho}_{f\circ g} = \rho_f$. Then*

$$\liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,f)} \le \frac{\overline{\sigma}_{f\circ g}}{\sigma_f} \le \limsup_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,f)}.$$

Proof. For arbitrary positive ε and for all sufficiently large values of r, we obtain that

$$\log T(r, f\circ g) \le (\overline{\sigma}_{f\circ g} + \varepsilon)\, r^{\overline{\rho}_{f\circ g}} \tag{9}$$

and

$$T(r,f) \le (\sigma_f + \varepsilon)\, r^{\rho_f}. \tag{10}$$

Also for a sequence of values of r tending to infinity we get

$$\log T(r, f\circ g) \ge (\overline{\sigma}_{f\circ g} - \varepsilon)\, r^{\overline{\rho}_{f\circ g}} \tag{11}$$

and

$$T(r,f) \ge (\sigma_f - \varepsilon)\, r^{\rho_f}. \tag{12}$$

Now combining (9) and (12) it follows for a sequence of values of r tending to infinity,

$$\frac{\log T(r, f\circ g)}{T(r,f)} \le \frac{(\overline{\sigma}_{f\circ g} + \varepsilon)\, r^{\overline{\rho}_{f\circ g}}}{(\sigma_f - \varepsilon)\, r^{\rho_f}}. \tag{13}$$

In view of condition (vi) and as $\varepsilon\ (> 0)$ is arbitrary we obtain from (13) that

$$\liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,f)} \le \frac{\overline{\sigma}_{f\circ g}}{\sigma_f}. \tag{14}$$

Also from (10) and (11) we get for a sequence of values of r tending to infinity,

$$\frac{\log T(r, f\circ g)}{T(r,f)} \ge \frac{(\overline{\sigma}_{f\circ g} - \varepsilon)\, r^{\overline{\rho}_{f\circ g}}}{(\sigma_f + \varepsilon)\, r^{\rho_f}}. \tag{15}$$

Since $\varepsilon\,(> 0)$ is arbitrary and $\overline{\rho}_{f\circ g} = \rho_f$ we obtain from (15) that

$$\limsup_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,f)} \ge \frac{\overline{\sigma}_{f\circ g}}{\sigma_f}. \tag{16}$$

Thus the theorem follows from (14) and (16). □

Remark 3. The sign of equality in Theorem 3 cannot be removed which is evident from the following example.

Example 2. Let $f = g = \exp z$. So

$$\rho_f = \rho_g = 1,\ \rho_{f\circ g} = \infty,\ \overline{\rho}_{f\circ g} = 1,\ \sigma_f = \frac{1}{\pi}$$

and

$$\overline{\sigma}_{f\circ g} = 1\ .$$

Also

$$T(r,f) = \frac{r}{\pi} \textit{ and } T(r,f\circ g) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\ .$$

Therefore

$$\liminf_{r\to\infty} \frac{\log T(r,f\circ g)}{T(r,f)} = \pi = \frac{\overline{\sigma}_{f\circ g}}{\sigma_f} = \limsup_{r\to\infty} \frac{\log T(r,f\circ g)}{T(r,f)}\ .$$

In the line of Theorem 3 we may state the following theorem without proof.

Theorem 4. *Let f be meromorphic and g be transcendental entire with (i) $0 < \rho_g < \infty$, (ii) $0 < \sigma_g < \infty$, (iii) $\rho_{f\circ g} = \infty$, (iv) $0 < \overline{\rho}_{f\circ g} < \infty$, (v) $0 < \overline{\sigma}_{f\circ g} < \infty$ and (vi) $\overline{\rho}_{f\circ g} = \rho_g$. Then*

$$\liminf_{r\to\infty} \frac{\log T(r,f\circ g)}{T(r,g)} \le \frac{\overline{\sigma}_{f\circ g}}{\sigma_g} \le \limsup_{r\to\infty} \frac{\log T(r,f\circ g)}{T(r,g)}.$$

References

[1] S.K. Datta, A. Jha, On the definition of type of a meromorphic function of order zero or infinity, *International Journal of Mathematical Analysis*, Accepted.

[2] F. Gross, *Factorization of Meromorphic Functions*, U.S. Government Printing Office, Washington, D.C. (1972).

[3] W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[4] G. Valiron, *Lectures on the General Theory of Integral Functions*, Chelsea Publishing Company (1949).