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FIXED POINT THEOREMS FOR TWO CLASSES OF MAPPINGS IN COMPACT METRIC SPACES

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Abstract: In this paper sufficient conditions of the existence of fixed points for two classes of nonlinear mappings are established in compact metric spaces, and sufficient and necessary conditions of a continuous mapping to possesses fixed points are given in compact metric spaces.

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1. Introduction and Preliminaries

Let f be a self mapping of a compact metric space (X, d) and ω denote the set of all nonnegative integers. In 1988, Jungck [2] defined $C_f = \{h : h : X \to$

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X and $fh = hf\}$ and proved the following result.

Theorem 1.1. (see [2]) *Let f and g be continuous commuting self mappings of a compact metric space (X, d). If $fx \neq gy$ implies that*

$$d(fx, gy) > d(hx, hy) \tag{1.1}$$

for some $h \in C_f \cap C_g$, then at least one of f or g has a fixed point in X.

Liu [4] extended Jungck's result and proved the following result.

Theorem 1.2. (see [4]) *Let f and g be continuous self mappings of a compact metric space (X, d) and $f(\cap_{n=1}^{\infty}(gf)^n X) = \cap_{n=1}^{\infty}(gf)^n X$. If f and g satisfy that*

$$\begin{aligned} d(fx, gy) > \inf\{&d(x, fx), d(y, fy), d(x, gx), \\ &d(y, gy), d(hx, hy) : h \in C_f \cap C_g\} \end{aligned} \tag{1.2}$$

for all $x, y \in X$ with $fx \neq gy$, then at least one of f or g has a fixed point in X.

Inspired by the results in [1]-[6], in this paper we introduce and study two classes of nonlinear mappings f and g as follows:

$$\begin{aligned} &d(fx, gy) + \min\{d(fx, y), d(x, gy)\} \\ &\quad > \inf\Big\{d(x, fx), d(y, fy), d(x, gx), d(y, gy), d(hx, hy), \\ &\quad \frac{d(x, fx)d(y, fy)}{d(fx, gy)}, \frac{d(x, fx)d(x, gx)}{d(fx, gy)}, \frac{d(x, fx)d(y, gy)}{d(fx, gy)}, \\ &\quad \frac{d(x, fx)d(hx, hy)}{d(fx, gy)}, \frac{d(y, fy)d(x, gx)}{d(fx, gy)}, \frac{d(y, fy)d(y, gy)}{d(fx, gy)}, \\ &\quad \frac{d(y, fy)d(hx, hy)}{d(fx, gy)}, \frac{d(x, gx)d(y, gy)}{d(fx, gy)}, \frac{d(x, gx)d(hx, hy)}{d(fx, gy)}, \\ &\quad \frac{d(y, gy)d(hx, hy)}{d(fx, gy)}, \frac{d(y, gy)[1 + d(y, fy)]}{1 + d(x, fx)}, \\ &\quad \frac{d(hx, hy)[1 + d(y, fy)]}{1 + d(x, fx)}, \frac{d(x, fx)[1 + d(x, y)]}{1 + d(y, gy)}, \\ &\quad \frac{d(y, fy)[1 + d(x, y)]}{1 + d(y, gy)}, \frac{d(hx, hy)[1 + d(x, y)]}{1 + d(y, gy)} : h \in C_f \cap C_g\Big\} \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &d(fx, gy) + \min\{d(f^{m+1}x, f^m y), d(g^m x, g^{m+1} y)\} \\ &\quad > \inf\Big\{d(hx, fhx), d(hy, fhy), d(hx, ghx), \end{aligned}$$

$$d(hy,ghy),d(hx,hy),\frac{d(f^px,f^py)d(g^qx,g^qy)}{d(fx,gy)},$$
$$\left.\frac{d(hx,fhx)d(kx,gkx)}{d(fx,gy)},\frac{d(hy,fhy)d(ky,gky)}{d(fx,gy)}:p,q\in\omega,\ h,k\in C_f\cap C_g\right\} \quad (1.4)$$

for all $x,y\in X$ with $fx\neq gy$ and some $m\in\omega$. Under certain conditions, we establish some fixed point theorems for the classes of mappings (1.3) and (1.4) in compact metric spaces. Our results extend and improve Theorems 1.1 and 1.2.

In order to obtain our results, we need the following result, which is due to Jungck [1] and Leader [3].

Lemma 1.1. (see [1], [3]) *Let f be a continuous self mapping of a compact metric space (X,d). If $B=\cap_{n=1}^{\infty}f^nX$, then*

(a) *B is compact*;

(b) $B=fB\neq\emptyset$;

(c) *$hB\subseteq B$ for $h\in C_f$.*

2. Main Results

Now we show the existence of fixed points for mappings (1.3) and (1.4) in compact metric spaces.

Theorem 2.1. *Let f and g be continuous self mappings of a compact metric space (X,d) and $f(\cap_{n=1}^{\infty}(gf)^nX)=\cap_{n=1}^{\infty}(gf)^nX$. If f and g satisfy* (1.3) *for all $x,y\in X$ with $fx\neq gy$, then at least one of f or g has a fixed point in X.*

Proof. Let $A=\cap_{n=1}^{\infty}(gf)^nX$. Using Lemma 1.1 and $fA=A$, we conclude that A is compact and $A=gfA=gA\neq\emptyset$. It follows that

$$hA\subseteq\cap_{n=1}^{\infty}h(gf)^nX=\cap_{n=1}^{\infty}(gf)^nhX \subseteq\cap_{n=1}^{\infty}(gf)^nX=A,\quad\forall h\in C_f\cap C_g. \quad (2.1)$$

Since f and g are continuous and A is compact, it follows that there exist $a,b\in A$ such that

$$d(a,fa)=\inf\{d(x,fx):x\in A\},\qquad d(b,gb)=\inf\{d(x,gx):x\in A\}. \quad (2.2)$$

Next we consider two possible cases below.

Case 1. Assume that

$$d(a, fa) \leq d(b, gb). \tag{2.3}$$

It follows from $gA = A$ that there exists some $c \in A$ such that $gc = a$. Suppose that $a \neq fa$, that is, $fgc \neq gc$. In view of (2.1)∼(2.3), we deduce that

$$\begin{aligned}
d(fgc, gc) &+ \min\{d(fgc, c), d(gc, gc)\} \\
&> \inf \Big\{ d(gc, fgc), d(c, fc), d(gc, ggc), d(c, gc), d(hgc, hc), \\
&\frac{d(gc, fgc)d(c, fc)}{d(fgc, gc)}, \frac{d(gc, fgc)d(gc, ggc)}{d(fgc, gc)}, \frac{d(gc, fgc)d(c, gc)}{d(fgc, gc)}, \\
&\frac{d(gc, fgc)d(hgc, hc)}{d(fgc, gc)}, \frac{d(c, fc)d(gc, ggc)}{d(fgc, gc)}, \frac{d(c, fc)d(c, gc)}{d(fgc, gc)}, \\
&\frac{d(c, fc)d(hgc, hc)}{d(fgc, gc)}, \frac{d(gc, ggc)d(c, gc)}{d(fgc, gc)}, \frac{d(gc, ggc)d(hgc, hc)}{d(fgc, gc)}, \\
&\frac{d(c, gc)d(hgc, hc)}{d(fgc, gc)}, \frac{d(c, gc)[1 + d(c, fc)]}{1 + d(gc, fgc)}, \\
&\frac{d(hgc, hc)[1 + d(c, fc)]}{1 + d(gc, fgc)}, \frac{d(gc, fgc)[1 + d(gc, c)]}{1 + d(c, gc)}, \\
&\frac{d(c, fc)[1 + d(gc, c)]}{1 + d(c, gc)}, \frac{d(hgc, hc)[1 + d(gc, c)]}{1 + d(c, gc)} : h \in C_f \cap C_g \Big\} \\
&= \inf \Big\{ d(a, fa), d(c, fc), d(a, ga), d(c, gc), d(ghc, hc), \\
&\frac{d(a, fa)d(c, fc)}{d(fa, a)}, \frac{d(a, fa)d(a, ga)}{d(fa, a)}, \frac{d(a, fa)d(c, gc)}{d(fa, a)}, \\
&\frac{d(a, fa)d(ghc, hc)}{d(fa, a)}, \frac{d(c, fc)d(a, ga)}{d(fa, a)}, \frac{d(c, fc)d(c, gc)}{d(fa, a)}, \\
&\frac{d(c, fc)d(ghc, hc)}{d(fa, a)}, \frac{d(a, ga)d(c, gc)}{d(fa, a)}, \frac{d(a, ga)d(ghc, hc)}{d(fa, a)}, \\
&\frac{d(c, gc)d(ghc, hc)}{d(fa, a)}, \frac{d(c, gc)[1 + d(c, fc)]}{1 + d(a, fa)}, \\
&\frac{d(ghc, hc)[1 + d(c, fc)]}{1 + d(a, fa)}, \frac{d(a, fa)[1 + d(gc, c)]}{1 + d(c, gc)}, \\
&\frac{d(c, fc)[1 + d(gc, c)]}{1 + d(c, gc)}, \frac{d(ghc, hc)[1 + d(gc, c)]}{1 + d(c, gc)} : h \in C_f \cap C_g \Big\} \\
&\geq \inf\{d(a, fa), d(b, gb), d(ghc, hc) : h \in C_f \cap C_g\} \geq d(a, fa),
\end{aligned}$$

which implies that

$$d(a, fa) = d(fgc, gc) > d(a, fa),$$

which is a contradiction. Hence $a = fa$.

Case 2. Assume that

$$d(a, fa) > d(b, gb). \tag{2.4}$$

It follows from $fA = A$ that there exists some $c \in A$ such that $fc = b$. Suppose that $b \neq gb$, that is, $gfc \neq fc$. In light of (2.1), (2.2) and (2.4), we arrive at

$$\begin{aligned}
&d(fc, gfc) + \min\{d(fc, fc), d(c, gfc)\} \\
&\quad > \inf\Big\{d(c, fc), d(fc, ffc), d(c, gc), d(fc, gfc), d(hc, hfc), \\
&\quad \frac{d(c, fc)d(fc, ffc)}{d(fc, gfc)}, \frac{d(c, fc)d(c, gc)}{d(fc, gfc)}, \frac{d(c, fc)d(fc, gfc)}{d(fc, gfc)}, \\
&\quad \frac{d(c, fc)d(hc, hfc)}{d(fc, gfc)}, \frac{d(fc, ffc)d(c, gc)}{d(fc, gfc)}, \frac{d(fc, ffc)d(fc, gfc)}{d(fc, gfc)}, \\
&\quad \frac{d(fc, ffc)d(hc, hfc)}{d(fc, gfc)}, \frac{d(c, gc)d(fc, gfc)}{d(fc, gfc)}, \frac{d(c, gc)d(hc, hfc)}{d(fc, gfc)}, \\
&\quad \frac{d(fc, gfc)d(hc, hfc)}{d(fc, gfc)}, \frac{d(fc, gfc)[1 + d(fc, ffc)]}{1 + d(c, fc)}, \\
&\quad \frac{d(hc, hfc)[1 + d(fc, ffc)]}{1 + d(c, fc)}, \frac{d(c, fc)[1 + d(c, fc)]}{1 + d(fc, gfc)}, \\
&\quad \frac{d(fc, ffc)[1 + d(c, fc)]}{1 + d(fc, gfc)}, \frac{d(hc, hfc)[1 + d(c, fc)]}{1 + d(fc, gfc)} : h \in C_f \cap C_g\Big\} \\
&\quad = \inf\Big\{d(c, fc), d(b, fb), d(c, gc), d(b, gb), d(hc, fhc), \\
&\quad \frac{d(c, fc)d(b, fb)}{d(b, gb)}, \frac{d(c, fc)d(c, gc)}{d(b, gb)}, \frac{d(c, fc)d(b, gb)}{d(b, gb)}, \\
&\quad \frac{d(c, fc)d(hc, fhc)}{d(b, gb)}, \frac{d(b, fb)d(c, gc)}{d(b, gb)}, \frac{d(b, fb)d(b, gb)}{d(b, gb)}, \\
&\quad \frac{d(b, fb)d(hc, fhc)}{d(b, gb)}, \frac{d(c, gc)d(b, gb)}{d(b, gb)}, \frac{d(c, gc)d(hc, fhc)}{d(b, gb)}, \\
&\quad \frac{d(b, gb)d(hc, fhc)}{d(b, gb)}, \frac{d(b, gb)[1 + d(b, fb)]}{1 + d(c, fc)}, \\
&\quad \frac{d(hc, fhc)[1 + d(b, fb)]}{1 + d(c, fc)}, \frac{d(c, fc)[1 + d(c, fc)]}{1 + d(b, gb)},
\end{aligned}$$

$$\frac{d(b,fb)[1+d(c,fc)]}{1+d(b,gb)}, \frac{d(hc,fhc)[1+d(c,fc)]}{1+d(b,gb)} : h \in C_f \cap C_g\Big\}$$
$$\geq \inf\{d(a,fa), d(b,gb), d(fhc,hc) : h \in C_f \cap C_g\} \geq d(b,gb),$$

which gives that

$$d(b,fb) = d(fc,gfc) > d(b,gb),$$

which is absurd. Hence $b = gb$. This completes the proof. □

Remark 2.1. Theorem 2.1 generalizes the corresponding results in [2] and [4].

Theorem 2.2. *Let f and g be continuous commuting self mappings of a compact metric space (X,d). Assume that there exists $m \in \omega$ satisfying* (1.4) *for all $x, y \in X$ with $fx \neq gy$. Then at least one of f or g has a fixed point in X.*

Proof. Let $A = \cap_{n=1}^{\infty}(fg)^n X$. It follows from Lemma 1.1 that A is a compact subset of X, $fgA = A \neq \emptyset$ and $hA \subseteq A$ for all $h \in C_f \cap C_g$. Note that

$$\begin{aligned} fA &= f \cap_{n=1}^{\infty} (fg)^n X \subseteq \cap_{n=1}^{\infty} f(fg)^n X \\ &= \cap_{n=1}^{\infty}(fg)^n fX \subseteq \cap_{n=1}^{\infty}(fg)^n X = A. \end{aligned} \tag{2.5}$$

Thus (2.5) ensures that

$$A = fgA \subseteq fA \subseteq A,$$

which implies that $A = fA$. Similarly we conclude also that $A = gA$. As in the proof of Theorem 2.1, we deduce easily that (2.1) and (2.2) hold. Assume that (2.3) is satisfied. It follows from $gA = A$ that there exists some $c \in A$ such that $gc = a$. Suppose that $a \neq fa$, that is, $fgc \neq gc$. In view of (2.1)∼(2.3) and (1.4), we get that

$$d(fgc,gc) + \min\{d(f^{m+1}gc, f^m c), d(g^m gc, g^{m+1}c)\}$$
$$> \inf\Big\{d(hgc,fhgc), d(hc,fhc), d(hgc,ghgc), d(hc,ghc), d(hgc,hc),$$
$$\frac{d(f^p gc, f^p c)d(g^q gc, g^q c)}{d(fgc,gc)}, \frac{d(hgc,fhgc)d(kgc,gkgc)}{d(fgc,gc)},$$
$$\frac{d(hc,fhc)d(kc,gkc)}{d(fgc,gc)} : p,q \in \omega,\ h,k \in C_f \cap C_g\Big\}$$
$$\geq \inf\Big\{d(ha,fha), d(hc,fhc), d(ha,gha), d(hc,ghc), d(ghc,hc),$$
$$d(gf^p c, f^p c), d(ha,fha), d(hc,fhc) : p,q \in \omega,\ h,k \in C_f \cap C_g\Big\} \geq d(fa,a),$$

which yields that

$$d(a, fa) = d(fgc, gc) > d(a, fa),$$

which is impossible. Hence $a = fa$.

Similarly, if (2.4) holds, we also infer that b is a fixed point of g. This completes the proof. □

It follows from Theorems 2.1 and 2.2 that

Corollary 2.1. *Let f and g be continuous self mappings of a compact metric space (X, d) and $f(\cap_{n=1}^{\infty}(gf)^n X) = \cap_{n=1}^{\infty}(gf)^n X$. If f and g satisfy that*

$$\begin{aligned} d(fx, gy) > \inf \Big\{ & d(x, fx), d(y, fy), d(x, gx), d(y, gy), d(hx, hy), \\ & \frac{d(x, fx)d(y, fy)}{d(fx, gy)}, \frac{d(x, fx)d(x, gx)}{d(fx, gy)}, \frac{d(x, fx)d(y, gy)}{d(fx, gy)}, \\ & \frac{d(x, fx)d(hy, hy)}{d(fx, gy)}, \frac{d(y, fy)d(x, gx)}{d(fx, gy)}, \frac{d(y, fy)d(y, gy)}{d(fx, gy)}, \\ & \frac{d(y, fy)d(hx, hy)}{d(fx, gy)}, \frac{d(x, gx)d(y, gy)}{d(fx, gy)}, \frac{d(x, gx)d(hx, hy)}{d(fx, gy)}, \\ & \frac{d(y, gy)d(hx, hy)}{d(fx, gy)}, \frac{d(y, gy)[1 + d(y, fy)]}{1 + d(x, fx)}, \\ & \frac{d(hx, hy)[1 + d(y, fy)]}{1 + d(x, fx)}, \frac{d(x, fx)[1 + d(x, y)]}{1 + d(y, gy)}, \\ & \frac{d(y, fy)[1 + d(x, y)]}{1 + d(y, gy)}, \frac{d(hx, hy)[1 + d(x, y)]}{1 + d(y, gy)} : h \in C_f \cap C_g \Big\} \end{aligned}$$

for all $x, y \in X$ with $fx \neq gy$, then at least one of f or g has a fixed point in X.

Corollary 2.2. *Let f and g be continuous commuting self mappings of a compact metric space (X, d). Assume that there exists $m \in \omega$ satisfying*

$$\begin{aligned} & d(fx, gy) + \min\{d(f^{m+1}x, f^m y), d(g^m x, g^{m+1}y)\} \\ & > \inf \Big\{ d(f^{n+1}x, f^n x), d(f^{n+1}y, f^n y), d(g^{n+1}x, g^n x), d(g^{n+1}y, g^n y), \\ & \qquad d(hx, hy), \frac{d(f^n x, f^{n+1}x)d(g^p x, g^{p+1}x)}{d(fx, gy)}, \\ & \qquad \frac{d(f^n y, f^{n+1}y)d(g^p y, g^{p+1}y)}{d(fx, gy)}, \frac{d(f^n x, f^n y)d(g^p x, g^p y)}{d(fx, gy)} : \\ & \qquad\qquad n, p \in \omega, \ h \in C_f \cap C_g \Big\}. \end{aligned}$$

Then at least one of f and g has a fixed point in X.

Theorem 2.3. *Let f be a continuous self mapping of a compact metric space (X,d). Then the following statements are equivalent:*

(a) *f has a fixed point in X;*

(b) *$fx \neq fy$ implies $d(fx,fy) > d(hx,hy)$ for some $h \in C_f$;*

(c) *$fx \neq fy$ implies*

$$\begin{aligned} d(fx,fy)+\min\{d(fx,y),d(x,fy)\} > \inf\Big\{ & d(x,fx),d(y,fy),d(hx,hy),\\ & \frac{d(x,fx)d(y,fy)}{d(fx,fy)},\frac{d^2(x,fx)}{d(fx,fy)},\frac{d(x,fx)d(hx,hy)}{d(fx,fy)},\\ & \frac{d^2(y,fy)}{d(fx,fy)},\frac{d(y,fy)d(hx,hy)}{d(fx,fy)},\frac{d(y,fy)[1+d(y,fy)]}{1+d(x,fx)},\\ & \frac{d(hx,hy)[1+d(y,fy)]}{1+d(x,fx)},\frac{d(x,fx)[1+d(x,y)]}{1+d(y,fy)},\\ & \frac{d(y,fy)[1+d(x,y)]}{1+d(y,fy)},\frac{d(hx,hy)[1+d(x,y)]}{1+d(y,fy)}: h\in C_f\cap C_g\Big\}. \end{aligned}$$

Proof. It follows from [1] that (a) and (b) are equivalent. It is clear that (b) implies (c). Taking $g = f$ in Theorem 2.1, we conclude that (c) implies (a). This completes the proof. □

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