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GENERAL SUBMANIFOLDS OF A KAEHLER MANIFOLD

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Abstract: In this paper we initiate the study of most general class of submanifolds of a Kaehler manifold which includes all existing classes of submanifolds (complex submanifolds, totally real submanifolds, CR-submanifolds, slant submanifolds). Such a submanifold M of a Kaehler manifold $\overline{M}$ has naturally defined operators ϕ, F, ψ and G (see Section 3 for the definition). First we study the basic properties of these operators for a general submanifold and latter we characterize submanifolds with parallel ϕ and show that essentially such submanifolds of a Kaehler manifold are CR-submanifold. We also give examples of submanifolds of a Kaehler manifold which have parallel ϕ and a submanifold on which the operator ϕ is not parallel.

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1. Introduction

There are two special types of geometries, geometry of Riemannian manifolds and its complex analogue the geometry of a Kaehler manifold. In that geom-

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etry of submanifolds of a Kaehler manifold has a special status because of the influence of the complex structure of the Kaehler manifold on the submanifold. Accordingly, there are various types of special submanifolds of a Kaehler manifold namely, complex submanifolds, totally real submanifolds, CR-submanifolds (this class includes both complex submanifolds and totally real submanifolds), Slant submanifolds. Complex submanifolds of a Kaehler manifold are those which are invariant under the complex structure of the Kaehler manifold and were first among the class of submanifolds of a Kaehler manifold studied with considerable interest by many mathematicians (cf. [38]). There is a large list of geometers who have contributed to the geometry of complex submanifold (cf. [13], [14], [17], [18], [23], [25], [27], [29], [36], [37], [40], [41], [42], [45], [46]). Among these authors, the work of Ros [41], [42] and Smyth [45], [46] are celebrated ones as not only they answered open questions, they have developed tools which are now applied in several studies of Differential Geometry. Next important class of submanifolds of a Kaehler manifold is the class of totally real submanifolds. A submanifold is said to be totally real submanifold if the complex structure of Kaehler manifold maps the tangent space of submanifold to normal subspace at each point of the submanifold. These submanifolds have also been quite extensively studied (cf. [4], [5], [6], [26], [30], [31], [32], [33], [39], [43]). There are special type of totally real submanifolds in which the tangent space and the normal space have the same dimension at each point of the submanifold, these totally real submanifolds are called Lagrangian submanifolds. Lagrangian submanifolds of a Kaehler manifolds have also been quite extensively studied (cf. [7], [8], [12], [24], [34], [44]). The complex submanifolds and totally real submanifolds arise very naturally in the geometry of submanifolds of a Kaehler manifold, however, Bejancu introduced a more general class of submanifolds called CR-submanifolds which includes both complex as well as totally real submanifolds (cf. [2], [3]). These submanifolds had been the subject of study for many geometers since its introductions (cf. [1], [19]-[22]) and so many papers in different journals. To give a geometric footing to the class of CR-submanifolds, Chen introduced a more general class than CR-submanifolds namely the class of Slant submanifolds (cf. [11]) and recently geometry of Slant submanifolds of Kaehler manifold is subject of interest to many mathematicians as it generalizes all existing classes of submanifolds of a Kaehler manifold (cf. [9], [10], [35]).

All these classes of submanifolds of a Kaehler manifold contain special types of submanifolds, however there is no much study being done on the submanifold of a Kaehler manifold without any condition on the complex structure of the Kaehler manifold. Such submanifold of a Kaehler manifold (without any condition on the impact of complex structure on the tangent spaces of the submanifold) we shall call as *general submanifold* of a Kaehler manifold. In literature we find only two references of this type of submanifolds (cf. [15], [16]), where the author studied extrinsic spheres in a Kaehler manifold and obtained their classification. Note that the submanifolds in all the existing classes of submanifolds, complex submanifolds, totally real submanifolds, CR-submanifolds, Slant submanifolds, are particular cases of general submanifolds and therefore it is important to study the geometry of these general submanifolds. In this paper we initiate the study of general submanifolds of a Kaehler manifold.

2. Preliminaries

Let M be an n-dimensional smooth manifold immersed into a $n+k=2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$ with Riemannian connection $\overline{\nabla}$ and the induced metric and connection on M be g and ∇ respectively. Then we have the following fundamental equations for the submanifold, namely

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X, Y \in \mathfrak{X}(M)\,, \tag{2.1}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad X \in \mathfrak{X}(M),\ N \in \Gamma(\upsilon), \tag{2.2}$$

where $\mathfrak{X}(M)$ is Lie-algebra of vector fields on M, $\Gamma(\upsilon)$ is the space of normal sections of the normal bundle υ of M, h is the second fundamental form, A_N is the Weingarten map with respect to $N \in \Gamma(\upsilon)$, $\nabla^{\perp}$ is the connection in the normal bundle υ. The Weingarten maps A_N are related to the second fundamental form h by

$$g(A_N X, Y) = g(h(X,Y), N), \quad X, Y \in \mathfrak{X}(M), \quad N \in \Gamma(\upsilon).$$

3. Basic Structure of General Submanifolds

For an n-dimensional submanifold M of a $n+k=2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$, define:

$$JX = \phi(X) + F(X), \quad JN = \psi(N) + G(N),$$

where $X \in \mathfrak{X}(M)$ and $N \in \Gamma(\upsilon)$, and $\phi(X) = (JX)^T$, $F(X) = (JX)^{\perp}$, $\psi(N) = (JN)^T$ and $G(N) = (JN)^{\perp}$, which define linear operators $\phi : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $F : \mathfrak{X}(M) \to \Gamma(\upsilon)$, $\psi : \Gamma(\upsilon) \to \mathfrak{X}(M)$ and $G : \Gamma(\upsilon) \to \Gamma(\upsilon)$ respectively. It is trivial implication of the definition that:

$$\phi^2(X) = -X - \psi(F(X)), \quad G^2(N) = -N - F(\psi(N)),$$

$$F(\phi(X)) = -G(F(X)), \quad \phi(\psi(N)) = -\psi(G(N)), \tag{3.1}$$

hold for $X \in \mathfrak{X}(M)$, $N \in \Gamma(\upsilon)$. Also

$$\begin{aligned} g(\phi(X), Y) &= g(JX - F(X), Y) = g(JX, Y) \\ &= -g(X, JY) = -g(X, \phi(Y)), \end{aligned} \tag{3.2}$$

similarly, we have

$$g(G(N_1), N_2) = -g(N_1, G(N_2))\,, \tag{3.3}$$

$$g(F(X), N) = -g(X, \psi(N))\,, \tag{3.4}$$

and

$$g(\psi(N), X) = -g(N, F(X)) \tag{3.5}$$

hold for $X, Y \in \mathfrak{X}(M)$ and $N, N_1, N_2 \in \Gamma(\upsilon)$.

Remark. Note that in general the rank of the operator ϕ need not be a constant. For the case rank $\phi =$ a constant $= \lambda$, then $\dim\ker\phi = n - \lambda$ and in this case we can define at each point $p \in M$,

$$D_p = T_pM \cap J_pT_pM,$$

where for $X \in \ker\phi$, we have $J_pX_p \in T_p^{\perp}M$ and consequently $X_p \notin D_p$, thus $\ker\phi_p = D_p^{\perp}$ is complementary subspace of D_p. Thus D_p and $D_p^{\perp}$ define smooth distributions D and $D^{\perp}$ such that $JD = D$ and $JD^{\perp} \subset \Gamma(\upsilon)$, making the submanifold a CR-submanifold of $\overline{M}$. In our work we are interested in the case where rank ϕ need not be a constant.

If we define the covariant derivatives $(D_XF)(Y)$ and $(D_X\psi)(N)$ for the operators $F : \mathfrak{X}(M) \to \Gamma(\upsilon)$ and $\psi : \Gamma(\upsilon) \to \mathfrak{X}(M)$ by

$$(D_XF)Y = \nabla_X^{\perp}F(Y) - F(\nabla_XY),$$

and

$$(D_X\psi)(N) = \nabla_X\psi(N) - \psi(\nabla_X^{\perp}N),$$

then we have the following which is trivial consequence of the Kaehler condition:

Lemma 3.1. *The operators ϕ, F, ψ and G obey*

$$(\nabla_X\phi)(Y) = A_{F(Y)}X + \psi(h(X,Y)),$$

$$(D_XF)(Y) = G(h(X,Y)) - h(X, \phi(Y)),$$

$$(D_X\psi)(N) = A_{G(N)}X - \phi(A_NX),$$

and

$$(\nabla_X^{\perp}G)(N) = -F(A_NX) - h(X, \psi(N)),$$

for $X, Y \in \mathfrak{X}(M)$ and $N \in \Gamma(\upsilon)$.

Proof. Using that $JY = \phi(Y) + F(Y)$ for $Y \in \mathfrak{X}(M)$, we have
$$\overline{\nabla}_X(JY) = \overline{\nabla}_X(\phi Y + FY).$$
Using Gauss and Weingarten formulas, we have
$$\begin{aligned}\overline{\nabla}_X(JY) &= \nabla_X(\phi Y) + h(X, \phi Y) - A_{FY}X + \nabla^{\perp}_X(FY)\\ &= (\nabla_X\phi)Y + \phi(\nabla_X Y) + h(X, \phi Y) - A_{FY}X + (D_X F)Y + F(\nabla_X Y),\end{aligned}$$
that is
$$\overline{\nabla}_X(JY) - \phi(\nabla_X Y) - F(\nabla_X Y) = (\nabla_X\phi)Y + h(X, \phi Y) - A_{FY}X + (D_X F)Y,$$
thus
$$\overline{\nabla}_X(JY) - J(\overline{\nabla}_X Y) + J(h(X, Y)) = (\nabla_X\phi)Y + h(X, \phi Y) - A_{FY}X + (D_X F)Y.$$
Since J is parallel, we get that
$$\psi(h(X, Y)) + G(h(X, Y)) = (\nabla_X\phi)Y + h(X, \phi Y) - A_{FY}X + (D_X F)Y.$$
Comparing the tangential and normal components of the both sides of this equation, we have
$$(\nabla_X\phi)Y = A_{FY}X + \psi(h(X, Y)),$$
$$(D_X F)Y = -h(X, \phi Y) + G(h(X, Y)).$$
Similarly, we have
$$\begin{aligned}\overline{\nabla}_X(JN) &= \overline{\nabla}_X(\psi N + GN) = \nabla_X(\psi N) + h(X, \psi N) - A_{GN}X + \nabla^{\perp}_X(GN)\\ &= (D_X\psi)N + \psi\nabla^{\perp}_X N + h(X, \psi N) - A_{GN}X + (\nabla^{\perp}_X G)N + G\nabla^{\perp}_X N,\end{aligned}$$
that is
$$\overline{\nabla}_X(JN) - \psi\nabla^{\perp}_X N - G\nabla^{\perp}_X N = (D_X\psi)N + h(X, \psi N) - A_{GN}X + (\nabla^{\perp}_X G)N,$$
$$\overline{\nabla}_X(JN) - J(\nabla^{\perp}_X N) = (D_X\psi)N + h(X, \psi N) - A_{GN}X + (\nabla^{\perp}_X G)N,$$
and we arrive at
$$\overline{\nabla}_X(JN) - J(\overline{\nabla}_X N) - J(A_N X) = (D_X\psi)N + h(X, \psi N) - A_{GN}X + (\nabla^{\perp}_X G)N,$$
that is
$$-\phi(A_N X) - F(A_N X) = (D_X\psi)N + h(X, \psi N) - A_{GN}X + (\nabla^{\perp}_X G)N.$$
Comparing the tangential and normal components, we get
$$(D_X\psi)N = A_{GN}X - \phi(A_N X),$$
$$(\nabla^{\perp}_X G)N = -F(A_N X) - h(X, \psi N)\,. \qquad \square$$

Note. If we define the operators
$$B : \psi \circ F : \mathfrak{X}(M) \to \mathfrak{X}(M),$$

and

$$C : F \circ \psi : \Gamma(\upsilon) \to \Gamma(\upsilon),$$

then it is easy to see that they are symmetric operators. Also, using (3.1) we see that B commutes with ϕ that is $B \circ \phi = \phi \circ B$ and that G commutes with C that is $G \circ C = C \circ G$. As a result of this we get $tr\ B \circ \phi = 0$ and $tr\ G \circ C = 0$. One important implication of this is that if X is an eigenvector of B with $BX = lX$, then ϕX is also an eigenvector of B with $B\phi X = l\phi X$.

4. Submanifolds with Parallel ϕ

In this section, we are interested in submanifolds with parallel ϕ, that is

$$\nabla_X \phi Y = \phi(\nabla_X Y),$$

where $X, Y \in \mathfrak{X}(M)$. Using Lemma 3.1, we immediately have

Lemma 4.1. *Let M be a submanifold of a Kaehler manifold $\overline{M}$. The operator ϕ is parallel if and only if*

$$A_{F(Y)}X = -\psi(h(X, Y)),$$

for $X, Y \in \mathfrak{X}(M)$.

Remark. Observe that if ϕ is parallel, then

$$A_{F(X)}Y = A_{F(Y)}X,$$

holds for $X, Y \in \mathfrak{X}(M)$ and that $g(h(X, Y), F(Z))$ is symmetric in X, Y, Z that is the following holds

$$g(h(X, Y), F(Z)) = g(h(Y, Z), F(X)) = g(h(X, Z), F(Y)),$$

where $X, Y, Z \in \mathfrak{X}(M)$.

Define operator B as the previous section, then from the equations (3.4) and (3.5), it is easy to see that

$$g(B(X), Y) = g(X, B(Y)), \quad X, Y \in \mathfrak{X}(M),$$

that is B is a symmetric tensor field of type $(1, 1)$.

Theorem 4.1. *Let M be an n-dimensional submanifold of a $(n+k) = 2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$. If the operator ϕ is parallel, then B is also parallel.*

Proof. We have for $X, Y \in \mathfrak{X}(M)$ that

$$\begin{aligned} \nabla_X(BY) &= \nabla_X \psi(F(Y)) = (D_X\psi)(F(Y)) + \psi(\nabla^{\perp}_X F(Y)) \\ &= (D_X\psi)(F(Y)) + \psi((D_X F)(Y)) + \psi(F(\nabla_X Y)), \end{aligned}$$

that is

$$(\nabla_X B)Y = (D_X\psi)(F(Y)) + \psi((D_X F)(Y)). \tag{4.1}$$

Using Lemma 3.1 in above equation, we get

$$(\nabla_X B)Y = A_{G(F(Y))}X - \phi A_{F(Y)}X + \psi(G(h(X,Y)) - h(X,\phi Y)). \tag{4.2}$$

Also, using equation (3.1), we get

$$(\nabla_X B)Y = -A_{F(\phi(Y))}X - \phi A_{F(Y)}X - \phi(\psi(h(X,Y)) - \psi(h(X,\phi Y))). \tag{4.3}$$

Finally, using Lemma 4.1 for parallel ϕ, we get

$$(\nabla_X B)(Y) = 0, \quad X, Y \in \mathfrak{X}(M).$$

that is B is parallel. □

Since the operator $B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is symmetric, it has real eigenvalues. For an eigenvalue l, define

$$D_l = \{X \in \mathfrak{X}(M) : B(X) = lX\}.$$

That is eigen distribution corresponding to eigenvalue l.

Lemma 4.2. *Let M be a submanifold with parallel ϕ. For $X \in D_l$, $X(l) = 0$.*

Proof. Using $(\nabla_X B)(Y) = 0$, for $X, Y \in D_l$, we have

$$0 = (\nabla_X B)(Y) = \nabla_X(BY) - B(\nabla_X Y) = \nabla_X(lY) - B\nabla_X Y.$$

Taking inner product with $Z \in D_l$, we get

$$0 = X(l)g(Y, Z)$$

that gives

$$X(l)\,\|X\|^2 = 0, \quad \forall X \in D_l$$

Since $\dim D_l \geq 1$, we get $X(l) = 0$ for $X \in D_l$. □

Lemma 4.3. *Let M be a submanifold with parallel ϕ. Then the distribution D_l is integrable and the leaves M_l of D_l are totally geodesic submanifolds of M.*

Proof. For $X, Y \in D_l$, using Lemma 4.2, we get

$$0 = (\nabla_X B)(Y) - (\nabla_Y B)(X) = l[X, Y] - B([X, Y]),$$

and this proves that D_l is involutive and consequently integrable. We also have

$$B(\nabla_X Y) = \nabla_X BY - (\nabla_X B)Y = l\nabla_X Y,$$

thus $\nabla_X Y \in D_l$, and this proves that the integral submanifolds, that is, leaves of D_l are totally geodesic submanifolds of M. □

Finally, we have the following using deRham's Theorem.

Theorem 4.2. *Let M be an n-dimensional submanifold of a $(n+k)=2m$-dimensional Kaehler manifold $(\overline{M},J,g)$. If the operator ϕ is parallel, then M is isometric to the product $M_1\times M_2\times\ldots\times M_k$, where each M_i is totally geodesic submanifold of M.*

Proof. Let $l_1,\ldots,l_k$ be the eigenvalues of B. Then each leaf M_k of the distribution

$$D_{l_k}=\{X\in\mathfrak{X}(M):B(X)=l_kX\},$$

is a totally geodesic submanifold by Lemma 4.3, then the rest of the proof follows from deRham's theorem. □

As a corollary of above theorem, we have

Corollary 4.1. *Let M be n-dimensional submanifold of a Kaehler manifold $(\overline{M},J,g)$. If ϕ is parallel and M is irreducible, then M is either a Kaehler manifold or a totally real submanifold.*

Proof. Since M is irreducible by Theorem 4.2, $B=lI$ for a constant l. Then equation (3.1), gives

$$\phi^2X=-X-\psi(F(X))=-X-B(X)=-X-lI(X)=-(1+l)X.$$

If $l=-1$, we get $\phi^2(X)=0$ and as ϕ is skew-symmetric, we get $\phi=0$, that is $JX=F(X)\in\Gamma(\upsilon)$, $\forall\ X\in\mathfrak{X}(M)$, that is M is totally real submanifold. If $l<-1$, then $\phi^2X=-\mu X$, with $\mu=1+l<0$, which would mean eigenvalues of ϕ are real with $\phi^2=a^2I$ and for a positive a^2 that cannot happens as ϕ is skew-symmetric. Thus $l>-1$ and in this case $\phi^2=-k^2I$ for a constant $k>0$. Define $J^{'}:\mathfrak{X}(M)\to\mathfrak{X}(M)$ by

$$J^{'}=\frac{1}{k}\phi,$$

then

$$J^{'}(J^{'}X)=\frac{1}{k}\phi(J^{'}X)=\frac{1}{k}\phi(\frac{1}{k}\phi(X))=\frac{1}{k^2}\phi^2(X)=\frac{1}{k^2}(-k^2X)=-X,$$

that is $J^{'2}=-I$, and as ϕ is parallel, $J^{'}$ is parallel with metric $g'=k^2g$, we see that $(M,J^{'},g')$ is a Kaehler manifold which is homothetic to (M,g). □

Let M be an n-dimensional submanifold of $(n+k)$-dimensional Kaehler manifold $(\overline{M},J,g)$. We continue our study of submanifold M with operator $\phi:\mathfrak{X}(M)\to\mathfrak{X}(M)$ parallel.

For a local orthonormal frame $\{e_1,e_2,...,e_n\}$ on M, we define smooth function $\|F\| : M \to R$ by

$$\|F\|^2 = \sum_{i=1}^{n} g(F(e_i), F(e_i)).$$

Then using

$$g(F(X), N) = -g(X, \psi(N)), \quad X \in \mathfrak{X}(M), \quad N \in \Gamma(\upsilon),$$

we obtain

$$\|F\|^2 = -trB,$$

where $B = \psi \circ F : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the symmetric operator. Similarly, we define

$$\|\phi\|^2 = \sum_{i} g(\phi(e_i), \phi(e_i)).$$

Lemma 4.4. *Let M be an n-dimensional submanifold of a Kaehler manifold $(\overline{M}, J, g)$ with parallel ϕ. Then $\|F\|^2$ is a constant and that $\|\phi\|^2 + \|F\|^2 = n$ holds, consequently $\|\phi\|^2$ is also a constant.*

Proof. Since by Theorem 4.1 ϕ is parallel implies B is parallel. Thus using $\|F\|^2 = -\sum_i g(e_i, B(e_i))$, we get for $X \in \mathfrak{X}(M)$ that

$$X(\|F\|^2) = -\sum_{i} g(e_i, (\nabla_X B)(e_i)) = 0,$$

that is $\|F\|^2$ is a constant.

Also, we have

$$n = \sum_{i=1}^{n} g(Je_i, Je_i) = \sum_{i=1}^{n} g(\phi(e_i) + F(e_i), \phi(e_i) + F(e_i))$$
$$= \sum_{i=1}^{n} g(\phi(e_i)), \phi(e_i)) + \sum_{i=1}^{n} g(F(e_i) + F(e_i)) = \|\phi\|^2 + \|F\|^2 .$$

Finally as $\|F\|^2$ is a constant, we get $\|\phi\|^2$ is also a constant. □

Recall that a symmetric tensor B of type $(1, 1)$ is called a Codazzi tensor if it satisfies

$$(\nabla_X B)Y = (\nabla_Y B)X, \quad X, Y \in \mathfrak{X}(M).$$

As Codazzi tensors are important in differential geometry, we are interested in the question under what condition the symmetric operator $B = \psi \circ F$ on the submanifold M is a Codazzi tensor.

Theorem 4.3. *Let M be a submanifold of a Kaehler manifold $(\overline{M}, J, g)$. Then the symmetric operator $B = \psi \circ F$ is a Codazzi tensor if the following* *hold*

$$A_{FX}Y = A_{FY}X \quad \text{and} \quad h(X, \phi Y) = h(\phi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Proof. Using equations (4.3) and (3.1), we get

$$(\nabla_X B)(Y) = -A_{F(\phi Y)}X - \phi A_{FY}X + \psi(G(h(X,Y))) - \psi(h(X, \phi Y)),$$

which gives

$$\begin{aligned}(\nabla_X B)(Y) - (\nabla_Y B)(X) &= A_{(F\circ\phi)(X)}Y - A_{(F\circ\phi)(Y)}X + \phi(A_{FX}Y - A_{FY}X) \\ &+ \psi(-h(X, \phi Y) + h(\phi X, Y)),\end{aligned}$$

if the conditions in the statement hold, then

$$(\nabla_X B)(Y) - (\nabla_Y B)(X) = A_{(F\circ\phi)X}Y - A_{(F\circ\phi)(Y)}X. \tag{4.4}$$

The condition $h(X\phi Y) = h(\phi X, Y)$ in the statement of the theorem is equivalent to

$$\phi \circ A_N = -A_N \circ \phi, \quad N \in \Gamma(\upsilon).$$

Thus using $A_{FX}Y = A_{FY}X$ and $\phi \circ A_N = -A_N \circ \phi$, we get that

$$\begin{aligned}A_{(F\circ\phi)(X)}Y &= A_{FY}\phi X = -\phi A_{FY}X = -\phi A_{FX}Y = A_{FX}\phi Y \\ &= A_{F(\phi Y)}X = A_{(F\circ\phi)(Y)}X,\end{aligned}$$

and consequently equation (4.4) gives that B is a Codazzi tensor. □

Theorem 4.4. *Let M be an n-dimensional connected submanifold of a Kaehler manifold with parallel ϕ. Then M is a CR-submanifold.*

Proof. Since ϕ is parallel, then by Lemma 4.4, $\|\phi\|^2$ is a constant, that is,

$$\sum_{i=1}^{n} g(\phi e_i, \phi e_i) = \text{constant} = \alpha. \tag{4.5}$$

If rank $\phi_p = r$ and rank $\phi_q = s$ for $p, q \in M$. Then at $p \in M$ we can choose orthonormal vectors $\{v_1, \dots, v_n\} \subset T_pM$ such that

$$\phi_p(v_i) \neq 0, \quad \text{for} \quad 1 \leq i \leq r \quad \text{and} \quad \phi_p(v_i) = 0 \quad \text{for} \quad i > r.$$

Let U be a neighborhood of p containing q and choose a local orthonormal frame $\{e_i, \dots, e_n\}$ on U such that $e_i/_p = v_i$. Then by (4.5), we have

$$\sum_{i=1}^{n} g(\phi e_i, \phi e_i)(p) = \alpha.$$

Note that,

$$X.g(\phi e_i, \phi e_i) = 2g((\nabla_X \phi)(e_i), e_i) = 0,$$

then each $g(\phi e_i, \phi e_i) =$ constant. Consequently if $g(\phi e_i, \phi e_i)(p) = 0$ for $i =$

$r+1,\ldots,n$. Hence $g(\phi e_i, \phi e_i)(q) = 0 \quad$ for $i = r+1,\ldots,n$. Thus

$$r \leq s.$$

Reversing the argument, we shall get $s \leq r$, consequently $s = r$. That means rank ϕ = constant if ϕ is parallel. Let

$$D^{\perp} = \{X \in \mathfrak{X}(M) : \phi(X) = 0\} = \ker \phi.$$

Then for $X \in D^{\perp}$ we shall have $JX = F(X) \in \Gamma(\upsilon)$. Consequently, $JD^{\perp} \subset \upsilon$. Moreover as rank$\phi$ = constant, then $\dim D^{\perp} = \dim \ker \phi$ = constant. Let D be the orthogonal complementary distribution to $D^{\perp}$, i.e.,

$$T_pM = D_p \oplus D_p^{\perp} \quad \text{for all} \quad p \in M.$$

Then we shall have for $X \in D$,

$$g(JX, Y) = 0 \quad \forall\, Y \in D^{\perp},$$

that is $JX \in D$ which implies $JD \subset D$ which gives that D is invariant under J with $\dim D = rank\phi$ = constant. This proves that M is a CR-submanifold of $(\overline{M}, J, g)$. □

Finally, we aim at giving examples of submanifolds of a Kaehler manifold with (i) parallel ϕ and (ii) ϕ not parallel.

Example 4.1. Consider the Kaehler manifold $(R^4, J, <,>)$, where J is the complex structure defined by

$$J(\frac{\partial}{\partial x^1}) = \frac{\partial}{\partial x^2}, \quad J(\frac{\partial}{\partial x^2}) = -\frac{\partial}{\partial x^1}, J(\frac{\partial}{\partial x^3}) = \frac{\partial}{\partial x^4}, \text{and} \quad J(\frac{\partial}{\partial x^4}) = -\frac{\partial}{\partial x^3},$$

where $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$, and $\frac{\partial}{\partial x^4}$ being coordinate vector fields on R^4 that is for each vector field

$$X = f^1\frac{\partial}{\partial x^1} + f^2\frac{\partial}{\partial x^2} + f^3\frac{\partial}{\partial x^3} + f^4\frac{\partial}{\partial x^4} \in \mathfrak{X}(R^4),$$

$$JX = -f^2\frac{\partial}{\partial x^1} + f^1\frac{\partial}{\partial x^2} - f^4\frac{\partial}{\partial x^3} + f^3\frac{\partial}{\partial x^4}, \tag{4.6}$$

and $<,>$ is the Euclidean metric on R^4. We denote by $\overline{\nabla}$ the Euclidean connection on R^4. Take $M = R^3$ and the embedding $f : M \to R^4$

$$f(x, y, z) = (y, x, 0, z).$$

Then we find the local orthonormal frame $\{e_1, e_2, e_3, N\}$ of R^4 where

$$e_1 = \frac{\partial}{\partial x^2}, \quad e_2 = \frac{\partial}{\partial x^1}, \quad e_3 = \frac{\partial}{\partial x^4} \quad \text{and} \quad N = \frac{\partial}{\partial x^3},$$

such that $\{e_1, e_2, e_3\}$ is local orthonormal frame on M. Let ∇ be the induced Riemannian connection on M. Then using properties of $\overline{\nabla}$, it is straight-forward to check that

$$\nabla_{e_i} e_j = 0, \quad i, j = 1, 2, 3, \tag{4.7}$$

and using (4.6), we find that

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1 \quad \text{and} \quad \phi(e_3) = 0 \tag{4.8}$$

and consequently, we get that

$$(\nabla_X \phi)(Y) = 0, \quad X, Y \in \mathfrak{X}(M),$$

this follows from (4.7) and (4.8) as each

$$X = l^1 e_1 + l^2 e_2 + l^3 e_3, \quad Y = \mu^1 e_1 + \mu^2 e_2 + \mu^3 e_3$$

Thus ϕ is parallel.

Next, we shall construct an example where ϕ is not parallel, that is M will not be a CR-submanifold.

Example 4.2. Consider 4-dimensional Euclidean space R^4 with Euclidean metric $<,>$. Then $(R^4, J, <,>)$ is a Kaehler manifold with the complex structure J defined by

$$J(\frac{\partial}{\partial x^1}) = \frac{\partial}{\partial x^2}, \quad J(\frac{\partial}{\partial x^2}) = -\frac{\partial}{\partial x^1}, \quad J(\frac{\partial}{\partial x^3}) = \frac{\partial}{\partial x^4}, \text{ and } J(\frac{\partial}{\partial x^4}) = -\frac{\partial}{\partial x^3},$$

where $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$, and $\frac{\partial}{\partial x^4}$ are the coordinate vector fields on R^4. It is easy to see that J is parallel with respect to the Euclidean connection $\overline{\nabla}$ on R^4, that is

$$\overline{\nabla}_X JY = J\overline{\nabla}_X Y,$$

holds for $X, Y \in \mathfrak{X}(R^4)$ the Lie-algebra of smooth vector fields on R^4.

Now consider the product $M = S^1 \times S^1$ of two copies of the unit circle S^1 and define

$$f : M \to R^4,$$

by

$$f(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) = (\cos\theta, \cos\varphi, \sin\theta, \sin\varphi),$$

where θ and φ are local coordinates of S^1 and S^1 respectively. Then it is straight forward to see that at $p = (\theta, \varphi) \in M$, differential df_p at $p \in M$ has the matrix respectively

$$df_p = \begin{bmatrix} -\sin\theta & 0 \\ 0 & -\sin\varphi \\ \cos\theta & 0 \\ 0 & \cos\varphi \end{bmatrix},$$

which has rank 2 at each $p \in M$; (as if $\sin\theta \sin\varphi = 0$, then $\cos\theta\cos\varphi \neq 0$ and vice-versa). Thus $f : M \to R^4$ is an immersion of M into R^4, that is, M a

2-dimensional submanifold of R^4. Choosing

$$\begin{aligned} e_1 &= -\sin\theta\frac{\partial}{\partial x^1}+\cos\theta\frac{\partial}{\partial x^3}, \quad e_2=-\sin\varphi\frac{\partial}{\partial x^2}+\cos\varphi\frac{\partial}{\partial x^4},\\ N_1 &= \cos\theta\frac{\partial}{\partial x^1}+\sin\theta\frac{\partial}{\partial x^3}, \quad N_2=\cos\varphi\frac{\partial}{\partial x^2}+\sin\varphi\frac{\partial}{\partial x^4}, \end{aligned} \tag{4.9}$$

we get a local orthonormal frame $\{e_1, e_2, N_1, N_2\}$ of R^4 such that $\{e_1, e_2\}$ is a local orthonormal frame on M with respect to the induced metric g as a submanifold of R^4 and that $\{N_1, N_2\}$ is local field of normal to M.

Let $\overline{e}_1 = \frac{\partial}{\partial\theta}$ and $\overline{e}_2 = \frac{\partial}{\partial\varphi}$ be the vector fields on the first and second copy of S^1 in $M = S^1 \times S^1$. Then we have

$$e_1 = df_{(\theta,\varphi)}(\overline{e}_1),$$

and

$$e_2 = df_{(\theta,\varphi)}(\overline{e}_2).$$

Next, we compute the values of ϕ at e_1 and e_2 respectively. Using

$$JX = \phi(X) + F(X), \quad X \in \mathfrak{X}(M),$$

and the equations (4.9), we get

$$Je_1 = J(-\sin\theta\frac{\partial}{\partial x^1}+\cos\theta\frac{\partial}{\partial x^3}) = (-\sin\theta\frac{\partial}{\partial x^2}+\cos\theta\frac{\partial}{\partial x^4}) \in \mathfrak{X}(R^4).$$

We can express it as

$$Je_1 = ae_1 + be_2 + cN_1 + dN_2,$$

where a, b, c and $d \in C^\infty(R^4)$. Then by (4.9)

$$\begin{aligned} Je_1 &= (-a\sin\theta + c\cos\theta)\frac{\partial}{\partial x^1} + (-b\sin\varphi + d\cos\varphi)\frac{\partial}{\partial x^2}\\ &+ (a\cos\theta + c\sin\theta)\frac{\partial}{\partial x^3} + (b\cos\varphi + d\sin\varphi)\frac{\partial}{\partial x^4}, \end{aligned}$$

equating the two values of Je_1, we conclude that

$$-a\sin\theta + c\cos\theta = 0,$$

$$a\cos\theta + c\sin\theta = 0,$$

$$-b\sin\varphi + d\cos\varphi = -\sin\theta,$$

and

$$b\cos\varphi + d\sin\varphi = \cos\theta.$$

Solving these equations, we get

$$a = 0, \quad b = \cos(\varphi - \theta), \quad c = 0 \quad \text{and} \quad d = \sin(\varphi - \theta),$$

that is

$$Je_1 = \cos(\varphi - \theta)e_2 + \sin(\varphi - \theta)N_2,$$

thus using

$$Je_1 = \phi(e_1) + F(e_1),$$

we arrive at

$$\phi(e_1) = \cos(\theta - \varphi)e_2,$$

and

$$F(e_1) = \sin(\varphi - \theta)N_2.$$

Similarly, using equation (4.9) we get

$$Je_2 = \sin\varphi\frac{\partial}{\partial x^1} - \cos\varphi\frac{\partial}{\partial x^3} \in \mathfrak{X}(R^4),$$

and consequently

$$Je_2 = -\cos(\varphi - \theta)e_1 + \sin(\varphi - \theta)N_1.$$

Thus we arrive at

$$\phi(e_2) = -\cos(\varphi - \theta)e_1,$$

and

$$F(e_2) = \sin(\varphi - \theta)N_1.$$

From these equations, we see that there are points where rank $\phi = 2$ (points where $\theta = \varphi$) and there are points where rank $\phi = 0$ (points where $\varphi - \theta = \pi/2$), thus for the submanifold M of the Kaehler manifold $(R^4, J, <,>)$, the rank ϕ is not a constant.

Now, we shall show that for this submanifold, ϕ is not parallel. Since the immersion $f : M \to R^4$ is local embedding, we have

$$\overline{\overline{\nabla}}_{e_1} e_1 = -\overline{e}_1(\sin\theta)\frac{\partial}{\partial x^1} + \overline{e}_1(\cos\theta)\frac{\partial}{\partial x^3}, \tag{4.10}$$

let ∇ be the Riemannian connection on M with respect to the induced metric g; then as

$$\overline{\overline{\nabla}}_{e_1} e_1 = \overline{a}e_1 + \overline{b}e_2 + \overline{c}N_1 + \overline{d}N_2, \tag{4.11}$$

where $\overline{a}, \overline{b}, \overline{c}$ and $\overline{d} \in C^\infty(R^4)$. Also using Gauss equation, we have

$$\overline{\nabla}_{e_1} e_1 = \nabla_{e_1} e_1 + h(e_1, e_1). \tag{4.12}$$

Inserting values of e_1, e_2, N_1 and N_2 into (4.11) and comparing with (4.10), we get

$$-\overline{a}\sin\theta + \overline{c}\cos\theta = -\overline{e}_1\sin\theta,$$

$$\overline{a}\cos\theta + \overline{c}\sin\theta = \overline{e}_1\cos\theta,$$

$$-\overline{b}\sin\varphi + \overline{d}\cos\varphi = 0,$$

and

$$\overline{b}\cos\varphi + \overline{d}\sin\varphi = 0.$$

Solving these equations and substituting in (4.11), we get

$$\overline{\nabla}_{e_1}e_1 = -N_1,$$

and from (4.12), we get that

$$\nabla_{e_1}e_1 = 0.$$

Similarly, computing for $\overline{\nabla}_{e_1}e_2$, $\overline{\nabla}_{e_2}e_1$ and $\overline{\nabla}_{e_2}e_2$, we get

$$\nabla_{e_1}e_2 = 0, \quad \nabla_{e_2}e_1 = 0 \quad \text{and} \quad \nabla_{e_2}e_2 = 0$$

(this is consistent with $R(e_1, e_2)e_1 = 0$ as M is flat torus).

Thus to compute $(\nabla_{e_1}\phi)(e_2)$, we have

$$\phi(e_2) = -\cos(\varphi - \theta)e_1,$$

and

$$\nabla_{e_1}\phi(e_2) = -\nabla_{e_1}(\cos(\varphi - \theta)e_1) = -\sin(\varphi - \theta)e_1,$$

and

$$\phi(\nabla_{e_1}e_2) = 0.$$

Thus

$$(\nabla_{e_1}\phi)(e_2) = -\sin(\varphi - \theta)e_1.$$

Similarly, we have $(\nabla_{e_1}\phi)(e_1) = -\sin(\theta - \varphi)e_2$, $(\nabla_{e_2}\phi)(e_1) = \sin(\theta - \varphi)e_2$, $(\nabla_{e_2}\phi)(e_2) = \sin(\varphi - \theta)e_1$. Since $\{e_1, e_2\}$ is a local orthonormal frame, we see that in general $(\nabla_{e_i}\phi)(e_j) \neq 0$, $i, j = 1, 2$, that is, there are points where

$$(\nabla_X\phi)(Y) \neq 0, \quad X, Y \in \mathfrak{X}(M).$$

that is ϕ is not parallel.

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