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ON A HYPERSURFACE WITH THE RICCI TENSOR SATISFYING CERTAIN CONDITIONS

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Abstract: The purpose of the present paper is to prove that a hypersurface with the nearly parallel Ricci tensor and a hypersurface with the cyclic parallel Ricci tensor under the assumption with constant mean curvature are locally symmetric.

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1. Introduction

Let $\tilde{M}^{n+1}(\tilde{c})$ be an $(n+1)$-dimensional space form of constant curvature $\tilde{c}$ (i.e. complete, simply connected Riemannian manifold with constant curvature, say, $\tilde{c}$). For each real number $\tilde{c}$, there is (up to isometry) exactly one space form in every dimension with sectional curvature $\tilde{c}$. The space forms of sectional curvature $\tilde{c}$ are denoted by $S^{n+1}(\tilde{c})$, R^{n+1} and H^{n+1} depending on whether $\tilde{c}$ is positive, zero or negative, respectively. $S^{n+1}(\tilde{c})$ is a Euclidean sphere of constant curvature $\tilde{c}$. R^{n+1} is a Euclidean space. H^{n+1} is a hyperbolic space of constant curvature $\tilde{c}$.

Let M^n be a hypersurface in a space form $\tilde{M}^{n+1}(\tilde{c})$. Let ∇ and S be the covariant differentiation on M^n and the Ricci tensor of M^n, respectively. P.J. Ryan [5] classified these hypersurfaces with regard to the parallel Ricci tensor, i.e., $\nabla S = 0$. He proved that if the Ricci tensor S of M^n is parallel and the mean

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curvature is constant, then M^n has the parallel second fundamental tensor, that is, M^n is locally symmetric and M^n is the product manifold of two space forms (see Theorem B).

The Ricci tensor S is called the *nearly parallel Ricci tensor* if M^n satisfies the condition of $(\nabla_X S)X = 0$ for any X tangent to M^n. The Ricci tensor S is also called the *cyclic Ricci parallel tensor* if M^n satisfies the condition of $(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$ for any X, Y and Z tangent to M^n.

The purpose of this paper is to classify hypersurfaces with the nearly parallel Ricci tensor and with the cyclic parallel Ricci tensor in a space form. We note that these condition are weaker than $\nabla S = 0$. We prove the following theorem:

Theorem 1. *Let M^n be a hypersurface of dimension $n \geq 2$ with constant mean curvature and the nearly parallel Ricci tensor in a space form $\tilde{M}^{n+1}(\tilde{c})$. Then M^n is parallel.*

Theorem 2. *Let M^n be a hypersurface of dimension $n \geq 2$ with constant mean curvature and the cyclic parallel Ricci tensor in a space form $\tilde{M}^{n+1}(\tilde{c})$. Then M^n is parallel.*

Corollary 3. *Let M^n be a hypersurface of dimension $n \geq 2$ with constant mean curvature. Then the following are equaivalent:* (1) *M has the nearly parallel Ricci tensor,* (2) *M has the cyclic parallel Ricci tensor,* (3) *M has the parallel Ricci tensor.*

2. Preliminaries

Let M^n be a hypersurface of dimension n in a space form $\tilde{M}_{n+1}(\tilde{c})$ of constant curvature $\tilde{c}$. For each point $x_0 \in M^n$, we choose an unit normal vector field ξ defined in a neighborhood $U(x_0)$ of x_0. Let $\tilde{\nabla}$ (resp. ∇) be the covariant differentiation on $\tilde{M}^{n+1}(\tilde{c})$ (resp. M^n). Then, for any vector fields X, Y tangent to M^n on $U(x_0)$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\xi, \tag{1}$$

$$\tilde{\nabla}_X \xi = -AX, \tag{2}$$

where g and A are the induced metric on M^n, and the (1, 1)-type symmetric tensor field called the *second fundamental form*, respectively.

Let R be the curvature tensor of M^n. Then, for any vector fields X, Y and Z on $U(x_0)$, we have the following (see [1]):

$$R(X,Y)Z = \tilde{R}(X,Y)Z + g(AY,Z)AX - g(AX,Z)AY$$
— Gauss equation, (3)

$$(\nabla_X A)Y = (\nabla_Y A)X$$
— Codazzi equation, (4)

where $\tilde{R}$ is the curvature tensor of $\tilde{M}^{n+1}(\tilde{c})$. Since $\tilde{M}^{n+1}(\tilde{c})$ is of constant curvature $\tilde{c}$, $\tilde{R}(X,Y)Z$ can be written as

$$\tilde{R}(X,Y)Z = \tilde{c}\{g(Y,Z)X - g(X,Z)Y\} \tag{5}$$

In particular, if Codazzi equation (4) satisfies

$$(\nabla_X A)Y = 0 \tag{6}$$

on a neighborhood of every point in M^n, then we say that *parallel second fundamental tensor* (see [2]).

Next, we denote the (0, 2)-type Ricci tensor of M^n by S. For any point x of $U(x_0)$, S is defined by

$$S(X,Y) = \sum_{i=1}^{n} g(R(X,e_i)e_i, Y), \tag{7}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space $T_x M^n$. Using Gauss equation (3) and the equation (5), we obtain

$$S(X,Y) = (n-1)\tilde{c}g(X,Y) + (\mathrm{trace}A)g(AX,Y) - g(A^2X,Y) \tag{8}$$

for any X tangent to M^n on $U(x_0)$. Setting $S(X,Y) = g(SX,Y)$, we can define the (1, 1)-type Ricci tensor of M^n. We also denote the (1, 1)-type Ricci tensor by the same symbol S.

We here recall the definitions of the nearly parallel Ricci tensor and the cyclic parallel Ricci tensor, again:

The Ricci tensor S is called the nearly parallel Ricci tensor if M^n satisfies $(\nabla_X S)X = 0$ for any X to M^n. The Ricci tensor S is called the cyclic parallel Ricci tensor if M^n satisfies $(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Z) = 0$. If S satisfies $(\nabla_X S)Y = 0$ for any X and Y tangent to M^n, then the Ricci tensor S is said to be parallel.

Now, we prepare the following results without proof.

Theorem A. (see Ryan, [5]) *If M^n is a hypersurface with parallel Ricci tensor in a space form $\tilde{M}^{n+1}(\tilde{c})$, then M^n is either the product manifold of two space forms or* rank$A = 2$ at all points of M^n.

Theorem B. (see Ryan, [5]) *Let M^n be a hypersurface with constant mean* *curvature of dimension $n > 1$ in a space form $\tilde{M}^{n+1}(\tilde{c})$ of constant curvature $\tilde{c}$. If the Ricci tensor of M^n is parallel, then M^n is locally symmetric and M^n is the product manifold of two space forms.*

3. The Nearly Parallel Ricci Tensor

Now, we prove the following theorem:

Theorem 1. *Let M^n be a hypersurface of dimension $n \geq 2$ with constant mean curvature in a space form $\tilde{M}_{n+1}(\tilde{c})$. If M^n has the nearly parallel Ricci tensor, then M^n is parallel.*

Proof. For each point x of $U(x_0)$, we choose an orthonormal basis of T_xM^n $\{e_1, \ldots, e_n\}$, for which A is diagonalized.

Since M^n has the nearly parallel Ricci tensor S, it follows that $(\nabla_{X+Y}S)(X+Y) = 0$ for any X, Y tangent to M^n on $U(x_0)$. From the equation (8), we obtain

$$S = (n-1)\tilde{c}I + (\mathrm{trace}A)A - A^2.$$

Therefore we have

$$\begin{aligned} - &\quad (\nabla_X A)AY - A(\nabla_X A)Y + (\mathrm{trace}A)(\nabla_X A)Y \\ = &\quad (\nabla_Y A)AX + A(\nabla_Y A)X - (\mathrm{trace}A)(\nabla_Y A)X. \end{aligned} \tag{9}$$

The following Codazzi equation (4) will be useful.

$$(\nabla_X A)Y = (\nabla_Y A)X.$$

Combining (9) with (4), we get

$$2A(\nabla_X A)Y = 2(\mathrm{trace}A)(\nabla_X A)Y - (\nabla_X A)AY - (\nabla_Y A)AX. \tag{10}$$

In order to proceed with the argument, we consider the distribution on $U(x_0)$ defined by

$$T_\lambda(x) = \{X \in T_xM^n | AX = \lambda X\}.$$

If M^n has k-distinct eigenvalues $\lambda_i, 1 \leq i \leq k$ at x, then we may assume that M^n has k-distinct eigenvalues on a neighborhood U of x. Let i_ℓ be the multiplicity of λ_i and $\{X_1, ..., X_{i_\ell}\}$ a basis of $T_{\lambda_i}(x)$. We extend $\{X_1, ..., X_{i_\ell}\}$ to vector fields on U and define vector fields

$$Y_t = (A - \lambda_1 I) \cdots (A \overset{\vee}{-} \lambda_i I) \cdots (A - \lambda_k I)X_t \text{ for } 1 \leq t \leq i_\ell,$$

where I denotes the identity transformation and $\vee$ means to neglect $A - \lambda_i I$. At x we have $Y_t = (\lambda_i - \lambda_1) \cdots (\lambda_i - \lambda_k)X_t$ for $1 \leq t \leq i_\ell$. Thus $Y_t, 1 \leq t \leq i_\ell$

are linearly independent at x and hence in U of x. At each point of U, we have

$$(A-\lambda_i)Y_t = 0 \text{ for } 1 \le t \le i_\ell.$$

Hence $Y_t, 1 \le t \le i_\ell$ forms a basis of T_{λ_i}. Therefore T_{λ_i}, $1 \le i \le k$ are differentiable.

Let $X \in T_\lambda, Y \in T_\mu$ and $Z \in T_\nu$. From (10) we have

$$A(\nabla_X A)Y = (\text{trace}A - \frac{\lambda}{2} - \frac{\mu}{2})(\nabla_X A)Y. \tag{11}$$

Hence, we obtain

$$(\text{trace}A - \frac{\lambda}{2} - \frac{\mu}{2} - \nu)g((\nabla_X Y, Z) = 0, \tag{12}$$

$$(\text{trace}A - \frac{\nu}{2} - \frac{\mu}{2} - \lambda)g((\nabla_Z A)Y, X) = 0. \tag{13}$$

Assume that $\lambda \ne \mu, \mu \ne \nu$ and $\nu \ne \lambda$. Noting that

$$g((\nabla_X A)Y, Z) = g((\nabla_Z A)Y, X),$$

if we suppose that $g((\nabla_X A)Y, Z) \ne 0$, then we have $\lambda = \nu$, which is a contradiction. Thus $g((\nabla_X A)Y, Z) = 0$. Hence we see that

$$(\nabla_X A)Y \in T_\lambda + T_\mu. \tag{14}$$

Moreover, let $X_1, X_2 \in T_\lambda, Y_1, Y_2 \in T_\mu$. From (10), we obtain

$$(\text{trace}A - \frac{\lambda}{2} - \frac{3\mu}{2})g((\nabla_{X_1} A)Y_1, Y_2) = 0.$$

Similarly, we have

$$(\text{trace}A - \mu - \lambda)g((\nabla_{Y_1} A)Y_2, X) = 0.$$

If $\text{trace}A - \frac{\lambda}{2} - \frac{3\mu}{2} = 0$ and $\text{trace}A - \mu - \lambda = 0$, then $\lambda = \mu$. This is contradiction. Hence, we know that $\text{trace}A - \frac{\lambda}{2} - \frac{3\mu}{2} \ne 0$ or $\text{trace}A - \mu - \lambda \ne 0$. Therefore, we get

$$g((\nabla_X A)Y_1, Y_2) = 0. \tag{15}$$

Also, from (10), we obtain

$$((\text{trace}A - \lambda - \mu)g(\nabla_{X_1} A)X_2, Y) = 0, \tag{16}$$

$$((\text{trace}A - \frac{3\lambda}{2} - \frac{\mu}{2})g(\nabla_{X_1} A)Y, X_2) = 0. \tag{17}$$

By the similar argument with the above we obtain

$$g((\nabla_{X_1} A)X_2, Y) = 0. \tag{18}$$

From (14), (15) and (18), if $\lambda \ne \mu$, then we have

$$(\nabla_X A)Y = 0.$$

If $\lambda \ne \frac{\text{trace}A}{2}$, then $(\nabla_{X_1} A)X_2 = 0$ for $X_1, X_2 \in T_\lambda$, noting that

$$\begin{aligned} & (\nabla_{X_1}A)(\nabla_{X_1}A)X_2 \in T_\lambda \\ = \quad & (X_1((\text{trace}A) - 2\lambda))(\nabla_{X_1}A)X_2. \end{aligned}$$

Finally, if $\lambda = \frac{\text{trace}A}{2}$, we construct the geodesic γ through x with initial tangent vector X_1 and we extend X_2 by parallel translation along γ. Now,

$$\nabla_{X_1}(A^2X_2 - (\text{trace}A)AX_2) = (A^2 - (\text{trace}A)A)\nabla_{X_1}X_2.$$

But $\nabla_{X_1}X_2 = 0$ along γ. We conclude that $A^2X_2 - (\text{trace}A)AX_2$ is parallel along γ. The value of this vector at x is $\frac{(\text{trace}A)^2}{4}X_2 - (\text{trace}A)(\frac{\text{trace}A}{2})X_2 = -\frac{(\text{trace}A)^2}{4}X_2$. But the vector $-\frac{(\text{trace}A)^2}{4}X_2$ is also parallel γ. Hence $A^2X_2 - (\text{trace}A)AX_2 = -\frac{(\text{trace}A)^2}{4}X_2$ all along γ. This means that

$$(A - \frac{(\text{trace}A)}{2}I)^2X_2 = 0 \text{ along } \gamma.$$

Again, since $(A - \frac{\text{trace}A}{2}I)$ is symmetric, we have that $AX_2 = \frac{\text{trace}A}{2}X_2$ along γ. Hence, along γ,

$$(\nabla_{X_1}A)X_2 = \nabla_{X_1}(AX_2) - A\nabla_{X_1}X_2 = \nabla_{X_1}(\frac{\text{trace}A}{2}X_2) - 0 = 0.$$

We have shown that $(\nabla_X A)Y = 0$ for any pair of principal vectors X and Y at any point $x \in M$. Since the principal vectors span the tangent space, we have shown that $\nabla A = 0$. This completes the proof of Theorem 1. □

4. The Cyclic Parallel Ricci Tensor

Now, we prove the following theorem.

Theorem 2. *Let M^n be a hypersurface of dimension $n \geq 2$ with constant mean curvature in a space form $\tilde{M}_{n+1}(\tilde{c})$. If M^n has the cyclic Ricci symmetric tensor, then M^n is parallel.*

Proof. For each point x of $U(x_0)$, we choose an orthonormal basis of T_xM^n $\{e_1, \ldots, e_n\}$, for which A is diagonalized.

Since M^n has the cyclic Ricci symmetric tensor S, it follows that

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$$

for any X, Y and Z tangent to M^n on $U(x_0)$. From the equation (8), we obtain

$$S(Y,Z) = (n-1)\tilde{c}g(Y,Z) + (\text{trace}A)g(AY,Z) - g(A^2Y,Z).$$

Therefore we have

$$\begin{aligned} &(\mathrm{trace}A)g((\nabla_X A)Y, Z) - g(A(\nabla_X A)Y + (\nabla_X A)AY, Z) \\ + \ &(\mathrm{trace}A)g((\nabla_Y A)Z, X) - g(A(\nabla_Y A)Z + (\nabla_Y A)AZ, X) \\ + \ &(\mathrm{trace}A)g((\nabla_Z A)X, Y) - g(A(\nabla_Z A)X + (\nabla_Z A)AX, Y) = 0. \end{aligned} \tag{19}$$

The following Codazzi equation (4) will be useful.

$$(\nabla_X A)Y = (\nabla_Y A)X.$$

Combining (19) with (4), we get

$$2A(\nabla_X A)Y = 3(\mathrm{trace}A)(\nabla_X A)Y - 2(\nabla_X A)AY - 2(\nabla_Y A)AX. \tag{20}$$

If M^n has k-distinct eigenvalues $\lambda_i, 1 \leq i \leq k$ at x, then we may assume that M^n has k-distinct eigenvalues on a neighborhood U of x. By the similar argument with Section 3 we know that T_{λ_i}, $1 \leq i \leq k$ are differentiable.

Let $X \in T_\lambda$ and $Y \in T_\mu$. Then, from (20), we obtain

$$A(\nabla_X A)Y = (\frac{3\mathrm{trace}A}{2} - \lambda - \mu)(\nabla_X A)Y. \tag{21}$$

Assume that $\lambda \neq \mu, \mu \neq \nu$ and $\nu \neq \lambda$. Let $X \in T_\lambda, Y \in T_\mu$ and $Z \in T_\nu$. From (20) we obtain

$$(\frac{3\mathrm{trace}A}{2} - \lambda - \mu - \nu)g((\nabla_X A)Y, Z) = 0.$$

If $\frac{3\mathrm{trace}A}{2} - \lambda - \mu - \nu \neq 0$ for any ν such that $\lambda \neq \nu$ and $\mu \neq \nu$, then we obtain

$$g((\nabla_X A)Y, Z) = 0.$$

Therefore, we have

$$(\nabla_X A)Y \in T_\lambda + T_\mu. \tag{22}$$

Assume that $\frac{3\mathrm{trace}A}{2} - \lambda - \mu - \nu = 0$ for some ν. Then we obtaion $(\nabla_X A)Y \in T_\nu$. Since

$$(X\mu)Y + (\mu I - A)\nabla_X Y \in T_\nu,$$

we have $X\mu = 0$, and, similarly, $X\nu = 0$. We construct the geodesic γ through x with initial tangent vector X and Y by parallel transformation along γ. Then we get

$$\nabla_X(A^2 Y - (\mu + \nu)AY) = (A^2 - (\mu + \nu)A)\nabla_X Y.$$

But $\nabla_X Y = 0$ along γ. We conclude that $A^2 Y - (\mu + \nu)AY$ is parallel along γ. The value of this vector at x is $\frac{(\mu+\nu)^2}{4}Y - (\mu + \nu)\frac{\mu+\nu)}{2}Y = -\frac{(\mu+\nu)^2}{4}Y$. But the vector $-\frac{(\mu+\nu)^2}{4}Y$ is also parallel along γ. Hence $A^2 Y - (\mu + \nu)AY = -\frac{(\mu+\nu)^2}{4}Y$ all along γ. This means that

$$(A - \frac{\mu + \nu}{2}I)^2 Y = 0 \ \text{ along } \ \gamma.$$

Again, since $A-\frac{\mu+\nu}{2}I$ is symmetric, we have that $AY=\frac{\mu+\nu}{2}Y$ along γ. Hence, along γ,

$$(\nabla_X A)Y = \nabla_X(AY) - A\nabla_X Y = \nabla_X(\frac{\mu+\nu}{2}Y) - 0 = \frac{\mu+\nu}{2}\nabla_X Y = 0.$$

Therefore, we see that $(\nabla_X A)Y = 0$.

Next, let $X_1, X_2 \in T_\lambda, Y_1, Y_2, Y_3 \in T_\mu$. From (20) we obtain

$$(\frac{3\mathrm{trace}A}{2} - 2\lambda - \mu)g((\nabla_{X_1}A)Y, X_2) = 0.$$

If $\frac{3\mathrm{trace}A}{2} - 2\lambda - \mu \neq 0$, then we obtain $g((\nabla_{X_1}A)Y, X_2) = 0$. Therefore

$$(\nabla_{X_1}A)X_2 \in T_\lambda.$$

Assume that $\frac{3\mathrm{trace}A}{2} - 2\lambda - \mu = 0$. Then $A(\nabla_X A)Y = \lambda(\nabla_X A)Y$. This implies that

$$(\nabla_X A)Y \in T_\lambda. \tag{23}$$

Hence we know that $g((\nabla_X A)Y_1, Y_2) = 0$, i.e., $(\nabla_{Y_1}A)Y_2 \in T_\mu$. On the other hand, we have

$$A(\nabla_{Y_1}A)Y_2 = (2\lambda - \mu)(\nabla_{Y_1}A)Y_2.$$

Thus we obtain $g((\nabla_{Y_1}A)Y_2, Y_3) = 0$. Therefore we have

$$(\nabla_{Y_1}A)Y_2 = 0, \quad \text{i.e.} \quad Y\mu = 0.$$

From trace$A$ = constant we see that $Y\lambda = 0$. Hence we have

$$g((\nabla_Y A)X_2, X_1) = 0. \tag{24}$$

Combining (23) with (24), we obtain

$$(\nabla_Y A)X = 0.$$

Thus it remains the case that $(\nabla_{X_1}A)X_2 \in T_\lambda$. Also, from (20), we have

$$A(\nabla_{X_1}A)X_2 = (\frac{3\mathrm{trace}A}{2} - 2\lambda)(\nabla_{X_1}A)X_2 \tag{25}$$

for $X_1, X_2 \in T_\lambda$.

If $\frac{\mathrm{trace}A}{2} \neq \lambda$, then we get

$$(\nabla_{X_1}A)X_2 = 0.$$

If $\lambda = \frac{\mathrm{trace}A}{2}$, then we construct the geodesic γ through x with initial tangent vector X_1 and we extend X_2 by parallel translation along γ. Now,

$$\nabla_{X_1}(A^2X_2 - (\mathrm{trace}A)AX_2) = (A^2 - (\mathrm{trace}A)A)\nabla_{X_1}X_2.$$

But $\nabla_{X_1}X_2 = 0$ along γ. We conclude that $A^2X_2 - (\mathrm{trace}A)AX_2$ is parallel along γ. The value of this vector at x is $\frac{(\mathrm{trace}A)^2}{4}X_2 - (\mathrm{trace}A)(\frac{\mathrm{trace}A}{2})X_2 = -\frac{(\mathrm{trace}A)^2}{4}X_2$. But the vector $-\frac{(\mathrm{trace}A)^2}{4}X_2$ is also parallel γ. Hence $A^2X_2 -$

$(\text{trace}A)AX_2 = -\frac{(\text{trace}A)^2}{4}X_2$ all along γ. This means that

$$(A - \frac{(\text{trace}A)}{2}I)^2 X_2 = 0 \text{ along } \gamma.$$

Again, since $(A - \frac{\text{trace}A}{2}I)$ is symmetric, we have that $AX_2 = \frac{\text{trace}A}{2}X_2$ along γ. Hence, along γ,

$$(\nabla_{X_1}A)X_2 = \nabla_{X_1}(AX_2) - A\nabla_{X_1}X_2 = \nabla_{X_1}(\frac{\text{trace}A}{2}X_2) - 0 = 0.$$

Thus we have shown that $(\nabla_X A)Y = 0$ for any pair of principal vectors X and Y at any point $x \in M$.

Since the principal vectors span the tangent space, we have shown that

$$(\nabla_X A)Y = 0.$$

This completes the proof of Theorem 2. □

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